

# ON PERMUTATION NUMERICAL SEMIGROUPS

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**ABSTRACT.** In this paper we introduce the notion of  $n$ -permutation numerical semigroup. While there are just three 2-permutation numerical semigroups, there are infinitely many  $n$ -permutation numerical semigroups if  $n > 2$ . We construct 16 families of 3-permutation numerical semigroups and one family of  $n$ -permutation numerical semigroups. Finally we present some experimental data, which seem to support a conjecture about the classification of 3-permutation numerical semigroups.

## 1. INTRODUCTION

A numerical semigroup  $G$  is a co-finite submonoid of the monoid of non-negative integers  $(\mathbb{N}, +, 0)$  (see [2] for a comprehensive monograph).

A set  $S \subseteq \mathbb{N}$  generates  $G$ , namely  $G = \langle S \rangle$ , if and only if the greatest common divisor  $\gcd(S)$  of the elements contained in  $S$  is equal to 1.

We can order the elements of  $G$  in such a way that

$$G = \{g_i : i \in \mathbb{N}\},$$

where any  $g_i$  is a non-negative integer,  $g_0 = 0$  and  $g_i < g_{i+1}$  for any  $i \in \mathbb{N}$ .

We can associate with  $G$  the strictly increasing sequence

$$g := (g_0, g_1, \dots) = (g_i)_{i \in \mathbb{N}}.$$

The sequence  $g$  can be reduced modulo a positive integer  $l$  defining

$$g \bmod l := (g_0 \bmod l, g_1 \bmod l, \dots) = (g_i \bmod l)_{i \in \mathbb{N}}.$$

If the value of  $l$  is not ambiguous, then we simply write

$$\overline{g} = (\overline{g_i})_{i \in \mathbb{N}}.$$

We give the following definition.

**Definition 1.1.** Let  $G := \{g_i : i \in \mathbb{N}\}$  be a numerical semigroup, where any  $g_i$  is a non-negative integer,  $g_0 = 0$  and  $g_i < g_{i+1}$  for any  $i \in \mathbb{N}$ .

Let  $n \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$  and  $S := \{g_1, g_2, \dots, g_n\}$ .

We say that  $G$  is a  *$n$ -permutation numerical semigroup* (briefly  *$n$ -permutation semigroup*) if  $G = \langle S \rangle$  and for any non-negative integer  $k$  the  $n$ -tuple

$$(\overline{g_{kn+1}}, \overline{g_{kn+2}}, \dots, \overline{g_{kn+n}})$$

contains exactly one representative for each residue class of  $\mathbb{Z}/n\mathbb{Z}$ .

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*Remark 1.2.* Rephrasing the definition, we can associate with any  $n$ -permutation semigroup  $G$  an infinite string  $g := (g_i)_{i \in \mathbb{N}^*}$  such that the modular string  $\bar{g} := (\bar{g}_i)_{i \in \mathbb{N}^*}$  is obtained through the concatenation of infinitely many strings of  $(\mathbb{Z}/n\mathbb{Z})^n$ , called  $n$ -permutations, each of which containing no repetitions.

We notice in passing that  $\bar{g}$  is ultimately periodic since  $G$  is a co-finite submonoid of  $(\mathbb{N}, +, 0)$ .

**Example 1.3.** Let  $G := \langle \{9, 14, 15, 16\} \rangle$ .

The increasing sequence of non-zero elements of  $G$  is

$$g = (9, 14, 15, 16, 18, 23, 24, 25, 27, 28, 29, 30, 31, 32, 33, 34, 36, \rightarrow),$$

which corresponds to the (modulo 4) sequence

$$\bar{g} = (\bar{1}, \bar{2}, \bar{3}, \bar{0}, \bar{2}, \bar{3}, \bar{0}, \bar{1}, \bar{3}, \bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{0}, \bar{1}, \bar{2}) \circ (\bar{0}, \bar{1}, \bar{2}, \bar{3}) \circ (\bar{0}, \bar{1}, \bar{2}, \bar{3}) \circ \dots,$$

namely

$$\bar{g} = (\bar{1}, \bar{2}, \bar{3}, \bar{0}) \circ (\bar{2}, \bar{3}, \bar{0}, \bar{1}) \circ (\bar{3}, \bar{0}, \bar{1}, \bar{2})^2 \circ (\bar{0}, \bar{1}, \bar{2}, \bar{3})^\infty.$$

The paper is organized as follows.

- In Section 3 we show that there are just three 2-permutation semigroups.
- In Section 4 we construct 16 families  $\{H_{i,k}\}_{i=1}^{16}$  of 3-permutation semigroups, namely we show that for any positive integer  $k$  the numerical semigroup, whose set of generators  $S$  is one of the following, is a 3-permutation semigroup.

Family	$S$
$H_{1,k}$	$\{3k, 3k+1, 6k-1\}$
$H_{2,k}$	$\{6k+1, 6k+2, 9k+3\}$
$H_{3,k}$	$\{6k+1, 9k+2, 9k+3\}$
$H_{4,k}$	$\{6k+1, 6k+3, 6k+5\}$
$H_{5,k}$	$\{6k+1, 12k-4, 12k\}$
$H_{6,k}$	$\{3k+1, 6k-1, 6k\}$
$H_{7,k}$	$\{3k+2, 3k+3, 3k+4\}$
$H_{8,k}$	$\{12k+2, 12k+4, 18k+3\}$
$H_{9,k}$	$\{3k+2, 6k+1, 6k+3\}$
$H_{10,k}$	$\{6k+3, 6k+5, 12k+4\}$
$H_{11,k}$	$\{6k+4, 6k+5, 9k+6\}$
$H_{12,k}$	$\{12k+4, 18k+3, 18k+5\}$
$H_{13,k}$	$\{6k+5, 9k+6, 9k+7\}$
$H_{14,k}$	$\{6k+5, 12k+4, 12k+6\}$
$H_{15,k}$	$\{12k+8, 12k+10, 18k+15\}$
$H_{16,k}$	$\{12k+8, 18k+13, 18k+15\}$

- In Section 5 we present some experimental data about 3-permutation semigroups. Driven by such experiments we conjecture that any 3-permutation semigroup  $G$  having multiplicity

$$m := \min\{x \in G : x > 0\}$$

at least equal to 12 belongs to one of the 16 families studied in Section 4. In the last part of the section we construct one family of  $n$ -permutation semigroups for any positive integer  $n \geq 3$ .

## 2. PRELIMINARIES

We introduce some notations we will use in the rest of the paper.

- If  $\{a, b\} \subseteq \mathbb{N}$  with  $a \leq b$ , then

$$\begin{aligned} [a, b] &:= \{x \in \mathbb{N} : a \leq x \leq b\}, \\ [a, b]_2 &:= \{x \in [a, b] : x \equiv a \pmod{2}\}. \end{aligned}$$

In particular  $[a, a] = [a, a]_2 = \{a\}$ .

- If  $A$  and  $B$  are two non-empty subsets of  $\mathbb{N}$  such that  $a < b$  (resp.  $a \leq b$ ) for any pair  $(a, b) \in A \times B$ , then we write  $A < B$  (resp.  $A \leq B$ ).
- If  $G$  is a numerical semigroup, then

$$F(G) := \max\{x : x \in \mathbb{Z} \setminus G\}$$

is called the Frobenius number of  $G$ .

- If  $n \in G \setminus \{0\}$ , then the set

$$\text{Ap}(G, n) := \{s \in G : s - n \notin G\}$$

is called the Apéry set of  $G$  with respect to  $n$ .

The following relation between the Apéry set and the Frobenius number will be used throughout the paper.

**Lemma 2.1.** *If  $G$  is a numerical semigroup and  $n \in G \setminus \{0\}$ , then*

$$F(G) = \max(\text{Ap}(G, n)) - n.$$

The following lemmas will be used repeatedly in the paper (see [1, Lemma 1] and [1, Section 5]).

**Lemma 2.2.** *Let  $k$  and  $e$  be two positive integers and*

$$S := \{a_i\}_{i=0}^k,$$

*where  $a_0$  is a positive integer and  $a_i := a_0 + ie$  for any  $i \in [1, k]$ .*

*Then  $x \in G := \langle S \rangle$  if and only if*

$$x = a_0q + er$$

*with  $\{q, r\} \subseteq \mathbb{N}$  and  $0 \leq r \leq kq$ .*

*Moreover*

$$F(G) = a_0 \left\lfloor \frac{a_0 - 2}{k} \right\rfloor + e(a_0 - 1).$$

The following fact, whose proof is immediate, will be used repeatedly in Lemmas 4.1 - 4.16.

**Lemma 2.3.** *Let  $(x_0, x_1, x_2)$  be a triple in  $\mathbb{N}^3$  such that one of the following holds:*

- $x_i = x_0 + i$  for  $i \in \{1, 2\}$ ;
- $x_i = x_0 + 2i$  for  $i \in \{1, 2\}$ .

*Then  $(\overline{x_0}, \overline{x_1}, \overline{x_2})$  is a 3-permutation.*

## 3. CLASSIFICATION OF 2-PERMUTATION SEMIGROUPS

The numerical semigroups

$$\begin{aligned} G_1 &:= \langle 1, 2 \rangle = \mathbb{N}, \\ G_2 &:= \langle 2, 3 \rangle = \{2, \rightarrow\}, \\ G_3 &:= \langle 3, 4 \rangle = \{3, 4, 6, \rightarrow\} \end{aligned}$$

are the only 2-permutation semigroups.

Indeed, one can easily check that  $G_1$ ,  $G_2$  and  $G_3$  are 2-permutation semigroups.

Now we suppose that  $G$  is a 2-permutation semigroup generated by  $S := \{a, b\}$ , where  $a$  and  $b$  are two coprime positive integers with  $2 \leq a < b$ .

According to the definition of 2-permutation semigroups the following hold:

- $a$  and  $b$  have different parity, namely  $b = a + h$  for some odd integer  $h$ ;
- $b < 2a$ .

We have that

$$\begin{aligned} S &= \{a, a + h\}, \\ 2S &= \{2a, 2a + h, 2a + 2h\}, \\ 3S &= \{3a, 3a + h, 3a + 2h, 3a + 3h\}, \\ 4S &= \{4a, \dots\}. \end{aligned}$$

First we suppose that  $a$  is odd and  $b$  is even.

We deal with different cases.

- *Case 1:*  $h < \frac{a}{3}$ .

The increasing sequence of the first 10 elements of  $G$  is

$$(a, a + h, 2a, 2a + h, 2a + 2h, 3a, 3a + h, 3a + 2h, 3a + 3h, 4a),$$

which reads (modulo 2) as follows:

$$(\overline{1}, \overline{0}, \overline{0}, \overline{1}, \overline{0}, \overline{1}, \overline{0}, \overline{1}, \overline{0}, \overline{0}).$$

Hence  $G$  cannot be a 2-permutation semigroup.

- *Case 2:*  $\frac{1}{3}a < h < \frac{1}{2}a$ .

The increasing sequence of the first 10 elements of  $G$  is

$$(a, a + h, 2a, 2a + h, 2a + 2h, 3a, 3a + h, 3a + 2h, 4a, 3a + 3h),$$

which reads (modulo 2) as follows:

$$(\overline{1}, \overline{0}, \overline{0}, \overline{1}, \overline{0}, \overline{1}, \overline{0}, \overline{1}, \overline{0}, \overline{0}).$$

Also in this case  $G$  cannot be a 2-permutation semigroup.

- *Case 3:*  $\frac{1}{2}a < h < a$ .

The increasing sequence of the first 8 elements of  $G$  is

$$(a, a + h, 2a, 2a + h, 3a, 2a + 2h, 3a + h, 4a),$$

which reads (modulo 2) as follows:

$$(\overline{1}, \overline{0}, \overline{0}, \overline{1}, \overline{1}, \overline{0}, \overline{0}, \overline{0}).$$

- *Case 4:*  $h = \frac{1}{3}a$ .

We notice that  $3 \mid a$  because  $h \in \mathbb{N}$ .

If  $a = 3$ , then  $G = G_3$ .

If  $a \neq 3$ , then  $h > 1$ . Therefore  $\gcd(a, b) > 1$  and  $G$  is not a numerical semigroup.

Now we suppose that  $a$  is even and  $b$  is odd.  
We distinguish two different cases.

- *Case 1:*  $h < \frac{a}{2}$ .

The increasing sequence of the first 6 elements of  $G$  is

$$(a, a + h, 2a, 2a + h, 2a + 2h, 3a),$$

which reads (modulo 2) as follows:

$$(\bar{0}, \bar{1}, \bar{0}, \bar{1}, \bar{0}, \bar{0}).$$

Hence  $G$  cannot be a 2-permutation semigroup.

- *Case 2:*  $\frac{a}{2} < h < a$ .

The increasing sequence of the first 6 elements of  $G$  is

$$(a, a + h, 2a, 2a + h, 3a, 2a + 2h),$$

which reads (modulo 2) as follows:

$$(\bar{0}, \bar{1}, \bar{0}, \bar{1}, \bar{0}, \bar{0}).$$

Hence  $G$  cannot be a 2-permutation semigroup.

- *Case 3:*  $h = \frac{a}{2}$ .

If  $a = 2$ , then  $G = G_2$ .

If  $a > 2$ , then  $h > 1$  and  $\gcd(a, b) > 1$ , namely  $G$  is not a numerical semigroup.

#### 4. SIXTEEN FAMILIES OF 3-PERMUTATION SEMIGROUPS

In the statements of Lemmas 4.1 - 4.16 we always suppose that  $G := \langle S \rangle$ , where

$$S := \{a_1, a_2, a_3\}$$

is a subset of  $\mathbb{N}^*$  such that

$$a_1 < a_2 < a_3.$$

We denote by  $g := (g_i)_{i=1}^\infty$  the increasing sequence of the positive elements in  $G$  and by  $\bar{g}$  its reduction modulo 3.

Moreover, if  $A$  is a subset of  $\mathbb{N}$ , we denote by  $g \cap A$  the subsequence of  $g$  formed by the elements of  $g$  belonging to  $A$  and by  $\bar{g} \cap \bar{A}$  its reduction modulo 3.

For example, if

$$g := (5, 7, 9, 10, 12, 14, \rightarrow),$$

$$A := \{7, 9, 10, 12\},$$

then

$$g \cap A = (7, 9, 10, 12),$$

$$\overline{g \cap A} = (\bar{1}, \bar{0}, \bar{1}, \bar{0}).$$

**Lemma 4.1.** *Let  $S := \{3k, 3k + 1, 6k - 1\}$  for some positive integer  $k$  and*

$$H_{1,k} := \cup_{i \in \mathbb{N}} (A_{i,k} \cup B_{i,k}),$$

where

$$\begin{aligned} A_{i,k} &:= [(2i)3k - i, (2i)3k + 2i], \\ B_{i,k} &:= [(2i+1)3k - i, (2i+1)3k + 2i + 1], \end{aligned}$$

for any  $i \in \mathbb{N}$ .

The following hold.

- (1)  $H_{1,k}$  is a submonoid of  $(\mathbb{N}, +, 0)$  containing  $S$ .
- (2)  $A_{i,k} < B_{i,k}$  for any  $i \in [0, k-1]$ .
- (3)  $B_{i,k} < A_{i+1,k}$  for any  $i \in [0, k-1]$ .
- (4)  $[(2(k-1)+1)3k - (k-1), \infty[ \subseteq H_{1,k}$ .
- (5)  $G = H_{1,k}$ .
- (6)  $H_{1,k}$  is a 3-permutation semigroup.

*Proof.* (1) Since  $\{3k, 3k+1\} = B_{0,k}$  and  $6k-1 \in A_{1,k}$  we have that  $S \subseteq H_{1,k}$ .

If  $\{x, y\} \subseteq H_{1,k}$ , then

$$\begin{aligned} j_1(3k) - \left\lfloor \frac{j_1}{2} \right\rfloor &\leq x \leq j_1(3k) + j_1, \\ j_2(3k) - \left\lfloor \frac{j_2}{2} \right\rfloor &\leq y \leq j_2(3k) + j_2, \end{aligned}$$

for some  $\{j_1, j_2\} \subseteq \mathbb{N}$ .

We notice that

$$\left\lfloor \frac{j_1}{2} \right\rfloor + \left\lfloor \frac{j_2}{2} \right\rfloor \leq \left\lfloor \frac{j_1 + j_2}{2} \right\rfloor.$$

Therefore

$$(j_1 + j_2)(3k) - \left\lfloor \frac{j_1 + j_2}{2} \right\rfloor \leq x + y \leq (j_1 + j_2)(3k) + (j_1 + j_2),$$

namely

$$x + y \in A_{\lfloor \frac{j_1 + j_2}{2} \rfloor, k} \cup B_{\lfloor \frac{j_1 + j_2}{2} \rfloor, k}.$$

(2) The assertion follows since

$$(2i+1)3k - i - [(2i)3k + 2i] = 3k - 3i \geq 3$$

for any  $i \in [0, k-1]$ .

(3) The assertion follows since

$$[2(i+1)3k - (i+1)] - [(2i+1)3k + 2i + 1] = 3k - 3i - 2 \geq 1$$

for any  $i \in [0, k-1]$ .

(4) Let  $b := (2(k-1)+1)3k - (k-1) \in B_{k-1,k}$ .

If  $x \in [b, \infty[$ , then there exists a set  $\{q, r\} \subseteq \mathbb{N}$  such that

$$\begin{cases} x - b = 3kq + r \\ 0 \leq r < 3k, \end{cases}$$

namely

$$\begin{cases} x = b + 3kq + r \\ 0 \leq r < 3k. \end{cases}$$

Since  $3k \in H_{1,k}$  and

$$b + r \in B_{k-1,k} \cup \{2k(3k) - k\} \subseteq B_{k-1,k} \cup A_{k,k},$$

we conclude that  $x \in H_{1,k}$ .

(5) Let  $x \in H_{1,k}$ . We show that  $x \in G$  dealing with two cases.

- If  $x \in [j(3k), j(3k) + j]$ , where  $j \in \{2i, 2i + 1\}$  for some  $i \in \mathbb{N}$ , then  $x \in \langle \{3k, 3k + 1\} \rangle$  in accordance with Lemma 2.2.
- If  $x \in [j(3k) - \lfloor \frac{j}{2} \rfloor, j(3k) - 1]$  for some  $j \in \mathbb{N}^*$ , then

$$x = j(3k) - t$$

for some  $t \in [1, \lfloor \frac{j}{2} \rfloor]$ . Therefore

$$x = (j - 2t)3k + t(6k - 1).$$

From (1) and (4) we deduce that  $H_{1,k}$  is a co-finite submonoid of  $(\mathbb{N}, +, 0)$ , namely  $H_{1,k}$  is a numerical semigroup.

Since

$$S \subseteq H_{1,k} \subseteq G,$$

we conclude that  $G = H_{1,k}$ .

(6) For any  $i \in \mathbb{N}$  we have that

$$|A_{i,k}| = 3i + 1,$$

$$|B_{i,k}| = 3i + 2.$$

Now let  $i \in [0, k - 1]$ . From (3) we deduce that

$$|B_{i,k} \cup A_{i+1,k}| = 6i + 6,$$

namely 3 divides  $|B_{i,k} \cup A_{i+1,k}|$ .

The sequence formed by the two greatest elements in  $B_{i,k}$  and by the smallest element of  $A_{i+1,k}$  reads as follows (modulo 3):

$$(\overline{2i}, \overline{2i + 1}, \overline{2i + 2}).$$

Therefore the elements of  $\overline{g \cap (B_{i,k} \cup A_{i+1,k})}$  are obtained via a concatenation of 3-permutations.

Finally we notice that

$$A_{k,k} \subseteq [(2(k - 1) + 1)3k - (k - 1), \infty[ \subseteq H_{1,k}.$$

Hence we conclude that  $H_{1,k}$  is a 3-permutation semigroup.  $\square$

**Lemma 4.2.** *Let  $S := \{6k + 1, 6k + 2, 9k + 3\}$  for some positive integer  $k$  and*

$$H_{2,k} := \{0\} \cup (\cup_{i \in \mathbb{N}^*} (A_{i,k} \cup B_{i,k})),$$

where

$$A_{i,k} := [(6k + 1)i, (6k + 1)i + i],$$

$$B_{i,k} := [(6k + 1)i + (3k + 2), (6k + 1)i + (3k + 2) + i - 1],$$

for any  $i \in \mathbb{N}^*$ .

The following hold.

- (1)  $H_{2,k}$  is a submonoid of  $(\mathbb{N}, +, 0)$  containing  $S$ .
- (2)  $A_{i,k} < B_{i,k}$  for any  $i \in [1, 3k + 1]$ .
- (3)  $B_{i,k} \leq A_{i+1,k}$  for any  $i \in [1, 3k]$ .
- (4)  $[(6k + 1)(3k) + (3k + 2), \infty[ \subseteq H_{2,k}$ .
- (5)  $G = H_{2,k}$ .
- (6)  $H_{2,k}$  is a 3-permutation semigroup.

*Proof.* (1) Since  $\{6k+1, 6k+2\} = A_{1,k}$  and  $9k+3 \in B_{1,k}$ , we have that  $S \subseteq H_{2,k}$ .

If  $\{x, y\} \subseteq H_{2,k}$ , then

$$\begin{aligned} (6k+1)i_1 + \varepsilon_1(3k+2) &\leq x \leq (6k+1)i_1 + \varepsilon_1(3k+1) + i_1, \\ (6k+1)i_2 + \varepsilon_2(3k+2) &\leq y \leq (6k+1)i_2 + \varepsilon_2(3k+1) + i_2, \end{aligned}$$

for some  $\{i_1, i_2\} \subseteq \mathbb{N}^*$  and  $\{\varepsilon_1, \varepsilon_2\} \subseteq \{0, 1\}$ .

If  $\varepsilon_1 = \varepsilon_2 = 0$ , then  $x + y \in A_{i_1+i_2,k}$ .

If  $\varepsilon_1 = \varepsilon_2 = 1$ , then  $x + y \in A_{i_1+i_2+1,k}$  because

$$\begin{aligned} x + y &\geq (6k+1)(i_1 + i_2) + 2(3k+2) \\ &= (6k+1)(i_1 + i_2 + 1) + 3 \end{aligned}$$

and

$$\begin{aligned} x + y &\leq (6k+1)(i_1 + i_2) + 2(3k+1) + (i_1 + i_2) \\ &= (6k+1)(i_1 + i_2 + 1) + (i_1 + i_2 + 1). \end{aligned}$$

If  $\varepsilon_1 \neq \varepsilon_2$ , then  $x + y \in B_{i_2+i_2,k}$ .

(2) The assertion follows since

$$(6k+1)i + (3k+2) - [(6k+1)i + i] = 3k+2-i \geq 1$$

for any  $i \in [1, 3k+1]$ .

(3) The assertion follows since

$$(6k+1)(i+1) - [(6k+1)i + (3k+2) + i - 1] = 3k - i \geq 0$$

for any  $i \in [1, 3k]$ .

(4) The proof is as in Lemma 4.1 (4).

(5) Let  $x \in H_{2,k} \setminus \{0\}$ . We show that  $x \in G$  dealing with two cases.

- If  $x \in A_{i,k}$  for some  $i \in \mathbb{N}^*$ , then  $x \in \langle \{6k+1, 6k+2\} \rangle$  in accordance with Lemma 2.2. Hence  $x \in G$ .
- If  $x \in B_{i,k}$  for some  $i \in \mathbb{N}^*$ , then

$$\begin{aligned} x &= (6k+1)i + (3k+2) + j \\ &= (6k+1)(i-1) + j + (9k+3) \end{aligned}$$

for some  $j \in [0, i-1]$ .

Since

$$(6k+1)(i-1) + j \in \langle \{6k+1, 6k+2\} \rangle,$$

we conclude that  $x \in G$ .

Therefore  $G = H_{2,k}$  as explained in Lemma 4.1.

(6) We notice that

$$H_{2,k} = \left( \bigcup_{i=1}^{3k} (A_{i,k} \cup B_{i,k}) \right) \cup [(6k+1)(3k) + (3k+2), \infty[$$

and  $B_{3k,k} \subseteq [(6k+1)(3k) + (3k+2), \infty[$ .

For any  $i \in \mathbb{N}^*$  we have that

$$\begin{aligned} |A_{i,k}| &= i+1, \\ |B_{i,k}| &= i. \end{aligned}$$

Moreover, if  $i_1, i_2$  and  $i_3$  are three consecutive positive integers such that

$$i_1 < i_2 < i_3 \leq 3k$$



where  $i_j \equiv j \pmod{3}$  for any  $j \in \{1, 2, 3\}$ , then

$$\sum_{j=1}^3 |A_{i_j, k} \cup B_{i_j, k}| \equiv 2(i_1 + i_2 + i_3) \equiv 0 \pmod{3},$$

namely  $|\cup_{j=1}^3 (A_{i_j, k} \cup B_{i_j, k})|$  is divisible by 3.

The sequence formed by the two greatest elements of  $A_{i_1}$  and the smallest element of  $B_{i_1}$  reads as follows (modulo 3):

$$(\overline{1}, \overline{2}, \overline{0}).$$

Since  $|A_{i_1, k} \cup B_{i_1, k}| \equiv 0 \pmod{3}$ , the elements of  $\overline{g \cap (A_{i_1, k} \cup B_{i_1, k})}$  are obtained via a concatenation of 3-permutations.

We notice that  $|A_{i_2, k}| \equiv 0 \pmod{3}$ .

The sequence formed by the two greatest elements of  $B_{i_2}$  and the smallest element of  $A_{i_3}$  reads as follows (modulo 3):

$$(\overline{1}, \overline{2}, \overline{0}).$$

Since  $|B_{i_2, k} \cup A_{i_3, k}| \equiv 0 \pmod{3}$ , the elements of  $\overline{g \cap (B_{i_2, k} \cup A_{i_3, k})}$  are obtained via a concatenation of 3-permutations.

Hence we can say that the elements of  $\overline{g \cap (\cup_{j=1}^3 (A_{i_j, k} \cup B_{i_j, k}))}$  are obtained by means of a concatenation of 3-permutations and the same holds for  $\overline{g \cap H_{2, k}}$ .  $\square$

**Lemma 4.3.** *Let  $S := \{a, b, c\}$ , where  $a := 6k + 1$ ,  $b := 9k + 2$  and  $c := 9k + 3$  for some positive integer  $k$ , and*

$$H_{3, k} := (S + \{0, a\}) \cup (\cup_{i \in \mathbb{N}^*} T_{i, k}),$$

where

$$T_{i, k} := A_{i, k} \cup B_{i, k} \cup C_{i, k} \cup D_{i, k} \cup E_{i, k} \cup F_{i, k} \cup G_{i, k} \cup I_{i, k} \cup J_{i, k}$$

with

$$\begin{aligned} A_{i, k} &:= \{(3i)a\}, \\ B_{i, k} &:= [(3i)a + 1, (3i)a + 3i], \\ C_{i, k} &:= [(3i)a + 3k + 1, (3i)a + 3k + 2 + 3(i - 1)], \\ D_{i, k} &:= A_{i, k} + \{a\}, \\ E_{i, k} &:= B_{i, k} + \{a\}, \\ F_{i, k} &:= [(3i)a + b, (3i)a + c + 3i], \\ G_{i, k} &:= D_{i, k} + \{a\}, \\ I_{i, k} &:= E_{i, k} + \{a\}, \\ J_{i, k} &:= F_{i, k} + \{a\}, \end{aligned}$$

for any  $i \in \mathbb{N}^*$ .

The following hold.

- (1)  $H_{3, k}$  is a submonoid of  $(\mathbb{N}, +, 0)$  containing  $S$ .
- (2) For any  $i \in [1, k - 1]$  we have that

$$\begin{aligned} A_{i, k} &< B_{i, k} < C_{i, k} < D_{i, k} < E_{i, k} < F_{i, k} < G_{i, k} < I_{i, k} < J_{i, k}, \\ J_{i, k} &< A_{i+1, k}, \end{aligned}$$

and  $A_{k,k} < B_{k,k} < C_{k,k} < D_{k,k}$ .

(3)  $[(3k+1)a, \infty[ \subseteq H_{3,k}$ .

(4)  $G = H_{3,k}$ .

(5)  $H_{3,k}$  is a 3-permutation semigroup.

*Proof.* (1) By definition of  $H_{3,k}$  we have that  $S \subseteq H_{3,k}$ .

Let  $\{i_1, i_2\} \subseteq \mathbb{N}^*$  and  $i_3 := i_1 + i_2$ .

If  $x \in T_{i_1,k}$  and  $y \in T_{i_2,k}$ , then  $x+y$  belongs to one of the rows 2-10, columns 2-10 of the following table.

	$A_{i_2,k}$	$B_{i_2,k}$	$C_{i_2,k}$	$D_{i_2,k}$	$E_{i_2,k}$	$F_{i_2,k}$	$G_{i_2,k}$	$I_{i_2,k}$	$J_{i_2,k}$
$A_{i_1,k}$	$A_{i_3,k}$	$B_{i_3,k}$	$C_{i_3,k}$	$D_{i_3,k}$	$E_{i_3,k}$	$F_{i_3,k}$	$G_{i_3,k}$	$I_{i_3,k}$	$J_{i_3,k}$
$B_{i_1,k}$		$B_{i_3,k}$	$C_{i_3,k}$	$E_{i_3,k}$	$E_{i_3,k}$	$F_{i_3,k}$	$I_{i_3,k}$	$I_{i_3,k}$	$J_{i_3,k}$
$C_{i_1,k}$			$E_{i_3,k}$	$F_{i_3,k}$	$F_{i_3,k}$	$I_{i_3,k}$	$J_{i_3,k}$	$J_{i_3,k}$	$B_{i_3+1,k}$
$D_{i_1,k}$				$G_{i_3,k}$	$I_{i_3,k}$	$J_{i_3,k}$	$A_{i_3+1,k}$	$B_{i_3+1,k}$	$C_{i_3+1,k}$
$E_{i_1,k}$					$I_{i_3,k}$	$J_{i_3,k}$	$B_{i_3+1,k}$	$B_{i_3+1,k}$	$C_{i_3+1,k}$
$F_{i_1,k}$						$B_{i_3+1,k}$	$C_{i_3+1,k}$	$C_{i_3+1,k}$	$E_{i_3+1,k}$
$G_{i_1,k}$							$D_{i_3+1,k}$	$E_{i_3+1,k}$	$F_{i_3+1,k}$
$I_{i_1,k}$								$E_{i_3+1,k}$	$F_{i_3+1,k}$
$J_{i_1,k}$									$I_{i_3+1,k}$

(2) All inequalities follow from the definition of the sets.

(3) The assertion holds because

$$D_{k,k} \cup E_{k,k} \cup F_{k,k} = [(3k)a + a, (3k)a + 12k + 3]$$

contains  $6k + 3$  consecutive integers.

Therefore, if  $x \in [(3k+1)a, \infty[$ , then

$$x = y + ha$$

for some  $y \in D_{k,k} \cup E_{k,k} \cup F_{k,k}$  and  $h \in \mathbb{N}$ .

(4) Let  $x \in H_{3,k}$ .

- If  $x \in (S + \{0, a\}) \cup A_{i,k} \cup D_{i,k} \cup G_{i,k}$  for some  $i \in \mathbb{N}^*$ , then we get immediately that  $x \in G$ .
- Let  $x \in B_{i,k}$  for some  $i \in \mathbb{N}^*$ . We notice that

$$B_{i,k} = [(2i)b - i + 1, (2i)b + 2i].$$

If  $x \in [(2i)b, (2i)b + 2i]$ , then  $x \in \langle \{b, c\} \rangle$  in accordance with Lemma 2.2.

If  $x = (2i)b - j$  for some  $j$  with  $1 \leq j \leq i - 1$ , then  $x \in G$  because

$$x = (2i)b - j = 2(i - j)b + (3j)a.$$

- Let  $x \in C_{i,k}$  for some  $i \in \mathbb{N}^*$ . We notice that  $C_{i,k} = [t, u]$ , where

$$t := (2i - 1)b + 2a - (i - 1),$$

$$u := (2i - 1)b + 2a + (2i - 1).$$

If  $x \in [(2i - 1)b + 2a, u]$ , then  $x \in \langle \{b, c\} \rangle + \{2a\}$  according to Lemma 2.2.

If  $x = (2i - 1)b + 2a - j$ , where  $1 \leq j \leq i - 1$ , then  $x \in G$  because

$$x = (2i - 2j - 1)b + 2a + (3j)a.$$

- Let  $x \in F_{i,k}$  for some  $i \in \mathbb{N}^*$ . We notice that

$$F_{i,k} = [(2i+1)b - i, (2i+1)b + (2i+1)].$$

If  $x \in [(2i+1)b, (2i+1)b + (2i+1)]$ , then  $x \in \langle \{b, c\} \rangle$  according to Lemma 2.2.

If  $x = (2i+1)b - j$ , where  $1 \leq j \leq i$ , then  $x \in G$  because

$$x = (2i - 2j + 1)b + (3j)a.$$

- If  $x \in E_{i,k} \cup I_{i,k} \cup J_{i,k}$ , then  $x \in G$  by the definition of the sets.

Hence we conclude that  $H_{3,k} = G$ .

(5) Let  $i \in [1, k]$ . Then

$$|A_{i,k}| \equiv 1 \pmod{3},$$

$$|B_{i,k}| \equiv 0 \pmod{3},$$

$$|C_{i,k}| \equiv 2 \pmod{3},$$

$$|A_{i,k} \cup B_{i,k} \cup C_{i,k}| \equiv 0 \pmod{3}.$$

The sequence formed by the element of  $A_{i,k}$  and the two smallest elements of  $B_{i,k}$  reads (modulo 3) as  $(\overline{0}, \overline{1}, \overline{2})$ .

The sequence formed by the greatest element of  $B_{i,k}$  and the two smallest elements of  $C_{i,k}$  reads (modulo 3) as  $(\overline{0}, \overline{1}, \overline{2})$ .

The remaining elements of  $\overline{g \cap C_{i,k}}$  are obtained through a concatenation of 3-permutations.

Therefore we conclude that the elements of  $\overline{g \cap (A_{i,k} \cup B_{i,k} \cup C_{i,k})}$  can be written as a concatenation of 3-permutations.

Since for any  $i \in [1, k-1]$  we have that

$$|D_{i,k} \cup E_{i,k} \cup F_{i,k}| = |G_{i,k} \cup I_{i,k} \cup J_{i,k}| = 3 + 6i,$$

using a similar argument we can prove that the elements of

$$\overline{g \cap (D_{i,k} \cup E_{i,k} \cup F_{i,k} \cup G_{i,k} \cup I_{i,k} \cup J_{i,k})}$$

can be obtained via a concatenation of 3-permutations.

Hence  $H_{3,k}$  is a 3-permutation semigroup. □

**Lemma 4.4.** *Let  $S := \{a, b, c\}$ , where*

$$a := 6k + 1, \quad b := a + 2, \quad c := a + 4,$$

*for some positive integer  $k$ , and*

$$t := \left\lfloor \frac{3}{2}k \right\rfloor, \quad \varepsilon := \begin{cases} 0 & \text{if } k \text{ is even,} \\ 1 & \text{if } k \text{ is odd.} \end{cases}$$

*Let*

$$H_{4,k} := (\cup_{i=1}^t A_{i,k}) \cup (\cup_{i=t+1}^{2t+\varepsilon} (B_{i,k} \cup C_{i,k})) \cup [(2t + \varepsilon + 1)a, \infty[,$$

*where*

$$A_{i,k} := [ia, ia + 4i]_2$$

*if  $i \in [1, t]$ , while*

$$B_{i,k} := [ia, ia + 2(2(i - t - 1) - \varepsilon)],$$

$$C_{i,k} := [ia + 2(2(i - t - 1) - \varepsilon + 1), ia + 2(2t + \varepsilon)]_2,$$

if  $i \in [t+1, 2t+\varepsilon]$ .

The following hold.

- (1)  $H_{4,k}$  is a submonoid of  $(\mathbb{N}, +, 0)$  containing  $S$ .
- (2) The following inequalities hold.
  - $A_{i,k} < A_{i+1,k}$  for any  $i \in [1, t-1]$ .
  - $A_{t,k} < B_{t+1,k}$  if  $k$  is even, while  $A_{t,k} < C_{t+1,k}$  and  $B_{t+1,k} = \emptyset$  if  $k$  is odd.
  - $B_{t+1,k} < C_{t+1,k}$  if  $k$  is even.
  - $B_{i,k} < C_{i,k}$  if  $i \in [t+2, 2t+\varepsilon]$ .
  - $C_{i,k} < B_{i+1,k}$  if  $i \in [t+1, 2t+\varepsilon-1]$ .
  - $C_{2t+\varepsilon,k} < [(2t+\varepsilon+1)a, \infty[$ .
- (3)  $G = H_{4,k}$ .
- (4)  $H_{4,k}$  is a 3-permutation semigroup.

*Proof.* (1) We notice that  $S = A_{1,k}$ .

Let  $x \in A_{i_1,k}$  and  $y \in A_{i_2,k}$  for some  $\{i_1, i_2\} \subseteq [1, t]$ . Then

$$\begin{aligned} x &= i_1 a + 2h_1 \quad \text{with } h_1 \in [0, 2i_1], \\ y &= i_2 a + 2h_2 \quad \text{with } h_2 \in [0, 2i_2]. \end{aligned}$$

- If  $i_1 + i_2 \leq t$ , then  $x + y \in A_{i_1+i_2,k}$ .
- If  $t < i_1 + i_2 \leq 2t + \varepsilon$  and  $h_1 + h_2 \leq 2t + \varepsilon$ , then

$$x + y \in B_{i_1+i_2,k} \cup C_{i_1+i_2,k}.$$

- If  $t < i_1 + i_2 < 2t + \varepsilon$  and  $h_1 + h_2 > 2t + \varepsilon$ , then

$$2(h_1 + h_2) > 4t + \varepsilon = 6k,$$

namely  $2(h_1 + h_2) \geq a$ .

Since

$$x + y = (i_1 + i_2 + 1)a + 2(h_1 + h_2) - a,$$

where

$$2(h_1 + h_2) - a \leq 4(i_1 + i_2) - (4t + 2\varepsilon) = 4(i_1 + i_2 + 1 - t - 1) - 2\varepsilon,$$

we conclude that

$$x + y \in B_{i_1+i_2+1,k}.$$

- If  $i_1 + i_2 > 2t + \varepsilon$  or  $i_1 + i_2 = 2t + \varepsilon$  and  $h_1 + h_2 > 2t + \varepsilon$ , then

$$x + y \in [(2t + \varepsilon + 1)a, \infty[.$$

(2) All inequalities follow from the definition of the sets.

(3) Let  $x \in H_{4,k}$ .

- If  $x \in A_{i,k}$  for some  $i \in [1, t]$ , then  $x \in G$  according to Lemma 2.2.
- If  $x \in C_{i,k}$  for some  $i \in [t+1, 2t+\varepsilon]$ , then  $x \in G$  according to Lemma 2.2.
- If  $x \in [(2t + \varepsilon + 1)a, \infty[$ , then  $x \in G$ .

Indeed, according to Lemma 2.2 we have that

$$\begin{aligned} F(G) &= \left\lfloor \frac{6k-1}{2} \right\rfloor a + 2(6k) = \left\lfloor \frac{4t-1+2\varepsilon}{2} \right\rfloor a + 2(6k) \\ &= (2t + \varepsilon - 1)a + 2(6k) = (2t + \varepsilon + 1)a - 2. \end{aligned}$$

- Let  $x \in B_{i,k}$  for some  $i \in [t+1, 2t+\varepsilon]$ .  
 If  $x = ia + 2h$  for some  $h \in [0, 2(i-t-1) - \varepsilon]$ , then  $h \leq 2i$  and  $x \in G$  according to Lemma 2.2.  
 If  $x = ia + \delta$  for some odd integer  $\delta \in [0, 4(i-t-1) - 2\varepsilon]$ , then

$$x = (i-1)a + \delta + a,$$

where

$$\delta \leq 4(i-t-1) - 2\varepsilon - 1.$$

We notice that

$$\frac{\delta + a}{2} = 2(i-t-1) - \varepsilon + 3k.$$

Since  $3k - 2t - \varepsilon \leq 0$ , we deduce that

$$\frac{\delta + a}{2} \leq 2(i-1)$$

and  $x \in G$  in accordance with Lemma 2.2.

Therefore we conclude that  $G = H_{4,k}$ .

- (4) If  $k$  is even, then  $t \equiv 0 \pmod{3}$ . Let  $i_1, i_2$  and  $i_3$  be three consecutive positive integers such that  $i_j \equiv j \pmod{3}$  for  $j \in [1, 3]$ . Then

$$|A_{i_1,k}| \equiv 0 \pmod{3},$$

$$|A_{i_2,k}| \equiv 2 \pmod{3},$$

$$|A_{i_3,k}| \equiv 1 \pmod{3},$$

and

$$|A_{i_1,k} \cup A_{i_2,k} \cup A_{i_3,k}| \equiv 0 \pmod{3}.$$

The sequence formed by the two greatest elements of  $A_{i_2,k}$  and the smallest element of  $A_{i_3,k}$  reads (modulo 3) as  $(\overline{2}, \overline{1}, \overline{0})$ .

The remaining elements of  $\overline{g \cap (A_{i_1,k} \cup A_{i_2,k} \cup A_{i_3,k})}$  are obtained through concatenations of 3-permutations.

For any  $i \in [t+1, 2t]$  we have that

$$|B_{i,k}| \equiv i \pmod{3},$$

$$|C_{i,k}| \equiv i-1 \pmod{3},$$

$$|B_{i,k} \cup C_{i,k}| \equiv 2i-1 \pmod{3}.$$

Moreover  $|[t+1, 2t]| = t \equiv 0 \pmod{3}$ .

Let  $i_1, i_2$  and  $i_3$  be three consecutive positive integers with  $i_j \equiv j \pmod{3}$  for  $j \in [1, 3]$ .

If  $x_{i_1}$  is the greatest element of  $B_{i_1,k}$  and  $y_{i_1} < z_{i_1}$  are the smallest elements of  $C_{i_1,k}$ , then

$$(\overline{x_{i_1}}, \overline{y_{i_1}}, \overline{z_{i_1}}) = (\overline{1}, \overline{0}, \overline{2}).$$

We have that  $|C_{i_1,k} \setminus \{y_{i_1}, z_{i_1}\}| \equiv 1 \pmod{3}$ . If  $x_{i_2}$  is the greatest element of  $C_{i_1,k}$  and  $y_{i_2} < z_{i_2}$  are the smallest elements of  $B_{i_2,k}$ , then

$$(\overline{x_{i_2}}, \overline{y_{i_2}}, \overline{z_{i_2}}) = (\overline{1}, \overline{2}, \overline{0}).$$

Since  $|B_{i_2,k} \setminus \{y_{i_2}, z_{i_2}\}| \equiv 0 \pmod{3}$ , we can concentrate on  $C_{i_2,k}$ .

If  $x_{i_3}$  is the greatest element of  $C_{i_2,k}$  and  $y_{i_3} < z_{i_3}$  are the smallest elements of  $B_{i_3,k}$ , then

$$(\overline{x_{i_3}}, \overline{y_{i_3}}, \overline{z_{i_3}}) = (\overline{2}, \overline{0}, \overline{1}).$$

We have that  $|B_{i_3,k} \setminus \{y_{i_3}, z_{i_3}\}| \equiv 1 \pmod{3}$ . If  $x_{i_4}$  is the greatest element of  $B_{i_3,k}$  and  $y_{i_4} < z_{i_4}$  are the smallest elements of  $C_{i_3,k}$ , then

$$(\overline{x_{i_4}}, \overline{y_{i_4}}, \overline{z_{i_4}}) = (\overline{2}, \overline{1}, \overline{0}).$$

We conclude that  $H_{4,k}$  is a 3-permutation semigroup when  $k$  is even.

Using similar arguments we can prove that  $H_{4,k}$  is a 3-permutation semigroup when  $k$  is odd.  $\square$

**Lemma 4.5.** *Let  $S := \{a, b, c\}$ , where*

$$a := 6k + 1, \quad b := 2a - 6, \quad c := 2a - 2$$

*for some integer  $k \geq 2$ .*

*Let*

$$H_{5,k} := \left( \bigcup_{i=0}^{k-1} (A_{i,k} \cup B_{i,k}) \right) \cup D_{k,k} \cup \left( \bigcup_{i=k+1}^{2k-1} (C_{i,k} \cup D_{i,k}) \right) \cup [(4k-1)a + 5, \infty[,$$

*where*

$$A_{i,k} := \{(2i+1)a, (2i+2)a - 6(i+1), (2i+2)a - 6(i+1) + 4\} \\ \cup [(2i+2)a - 6i, (2i+2)a - 2]_2,$$

$$B_{i,k} := \{(2i+2)a, (2i+3)a - 6(i+1), (2i+3)a - 6(i+1) + 4\} \\ \cup [(2i+3)a - 6i, (2i+3)a - 2]_2,$$

*for  $i \in [0, k-2]$ ,*

$$A_{k-1,k} := \{(2k-1)a, (2k)a - 6k, (2k)a - 6k + 4\} \\ \cup [(2k)a - 6k + 6, (2k)a - 2]_2,$$

$$B_{k-1,k} := \{(2k)a, (2k+1)a - 6k, (2k+1)a - 6k + 4\} \\ \cup [(2k+1)a - 6k + 6, (2k+1)a - 8]_2 \\ \cup [(2k+1)a - 6, (2k+1)a - 4] \cup [(2k+1)a - 2, (2k+1)a],$$

$$D_{k,k} := [(2k+1)a + 1, (2k+1)a + 1 + 6k - 8]_2 \\ \cup [(2k+2)a - 6, (2k+2)a - 4] \cup [(2k+2)a - 2, (2k+2)a],$$

*and*

$$C_{i,k} := [(2i)a + 1, (2i)a + 1 + 2(6k - 3i - 4)]_2 \\ \cup [(2i)a + 1 + 2(6k - 3i - 3), (2i)a + 3 + 2(6k - 3i - 3)] \\ \cup [(2i)a + 5 + 2(6k - 3i - 3), (2i+1)a],$$

$$D_{i,k} := [(2i+1)a + 1, (2i+1)a + 1 + 2(6k - 3i - 4)]_2 \\ \cup [(2i+1)a + 1 + 2(6k - 3i - 3), (2i+1)a + 3 + 2(6k - 3i - 3)] \\ \cup [(2i+1)a + 5 + 2(6k - 3i - 3), (2i+2)a],$$

*for  $i \in [k+1, 2k-1]$ .*

*The following hold.*

*(1) We have that*

$$C_{2k-1,k} = [(4k-2)a + 1, (4k-2)a + 3] \cup [(4k-2)a + 5, (4k-1)a],$$

$$D_{2k-1,k} = [(4k-1)a + 1, (4k-1)a + 3] \cup [(4k-1)a + 5, (4k)a].$$

(2)  $x \in G$  if and only if

$$x = (p + 2q)a + 4r - 6q,$$

where  $\{p, q, r\} \subseteq \mathbb{N}$  and  $0 \leq r \leq q$ .

(3)  $H_{5,k} \subseteq G$ .

(4)  $G \subseteq H_{5,k}$ .

(5) The following inequalities hold:

$$\begin{aligned} A_{i,k} &< B_{i,k} && \text{for any } i \in [0, k-1]; \\ B_{i,k} &< A_{i+1,k} && \text{for any } i \in [0, k-2]; \\ B_{k-1,k} &< D_{k,k}; \\ C_{i,k} &< D_{i,k} && \text{for any } i \in [k+1, 2k-1]; \\ D_{i,k} &< C_{i+1,k} && \text{for any } i \in [k+1, 2k-2]. \end{aligned}$$

(6)  $H_{5,k}$  is a 3-permutation semigroup.

*Proof.* (1) Both the assertions follow from the definition of the sets.

(2) We have that  $x \in G$  if and only if

$$x = pa + sb + tc$$

for some  $\{p, s, t\} \subseteq \mathbb{N}$ .

Since  $sb + tc \in \langle b, c \rangle$ , according to Lemma 2.2 we can write

$$x = pa + qb + 4r = (p + 2q)a + 4r - 6q,$$

where  $0 \leq r \leq q$ .

(3) We prove the assertion dealing with different cases.

- *Case 1:*  $x \in A_{i,k} \cup B_{i,k}$  with  $i \in [0, k-2]$ .

Since the case  $x \in B_{i,k}$  is very similar to the case  $x \in A_{i,k}$ , we discuss in detail just this latter one.

If  $x \in \{(2i+1)a, (2i+2)a - 6(i+1), (2i+2)a - 6(i+1) + 4\}$ , then  $x \in G$  by the characterization of the elements belonging to  $G$ .

Now suppose that  $x = (2i+2)a - 6i + 2h$  with  $h \in [0, 3i-1]$ . We notice that  $-6i + 2h$  can be written in the form

$$-6\tilde{i} + 4\tilde{h},$$

where  $\tilde{i} \in [1, i+1]$  and  $\tilde{h} \in [0, 2]$ . Hence

$$x = (2 + 2(i - \tilde{i}) + 2\tilde{i})a + 4\tilde{h} - 6\tilde{i},$$

where  $\tilde{i}$  and  $\tilde{h}$  are as above.

We have that  $\tilde{h} \leq \tilde{i}$  in the case that  $\tilde{i} \geq 2$ .

If  $\tilde{i} = 1$ , then  $\tilde{h} \leq 1$ . In fact, if  $\tilde{h} = 2$ , then  $x > (2i+2)a$  in contradiction with the fact that  $x < (2i+2)a$ .

- *Case 2:*  $x \in A_{k-1,k} \cup B_{k-1,k}$ .

The proof follows the same lines as in Case 1. We discuss in detail just the case

$$x \in [(2k+1)a - 6, (2k+1)a - 4] \cup [(2k+1)a - 2, (2k+1)a].$$

More precisely we show that any  $x$  in such a range belongs to  $G$ .

Obviously we have that  $(2k+1)a \in G$ . Therefore we deal with the remaining elements.

$$\begin{aligned} - & (2k+1)a - 6 = (1 + 2(k-1) + 2 \cdot 1)a - 6 \cdot 1. \\ - & (2k+1)a - 5 = (2(k+1))a - 6(k+1). \end{aligned}$$

- $$- (2k+1)a - 4 = (1 + 2(k-2) + 2 \cdot 2)a + 4 \cdot 2 - 6 \cdot 2.$$

$$- (2k+1)a - 2 = (1 + 2(k-1) + 2 \cdot 1)a + 4 \cdot 1 - 6 \cdot 1.$$

$$- (2k+1)a - 1 = (2(k+1))a + 4 \cdot 1 - 6(k+1).$$
- *Case 3:*  $x \in C_{i,k} \cup D_{i,k}$  with  $i \in [k+1, 2k-1]$  or  $x \in D_{k,k}$ .  
 We discuss in detail just the case  $x \in C_{i,k}$ .  
 If  $x \in C_{i,k}$  and  $x$  is odd, then

$$\begin{aligned} x &= (2i)a + 1 + 2h \\ &= (2i+1)a - 6k + 2h \end{aligned}$$

for some  $h \in [0, 3k]$ .

We can prove that  $x \in G$  as in Case 1. More precisely,

$$-6k + 2h = -6\tilde{k} + 4\tilde{h}$$

for some  $\tilde{k} \in [1, k+1]$  and  $\tilde{h} \in [0, 2]$ .

Since

$$\begin{aligned} (2i+1)a - 6k + 2h &= (2i+1)a - 6\tilde{k} + 4\tilde{h} \\ &= (1 + 2(i - \tilde{k}) + 2\tilde{k})a + 4\tilde{h} - 6\tilde{k}, \end{aligned}$$

we conclude that  $x \in G$ .

If  $x \in C_{i,k}$  and  $x$  is even, then

$$x = (2i)a + 2 + 2(6k - 3i - 3) \quad \text{or} \quad x \geq (2i)a + 6 + 2(6k - 3i - 3).$$

In the former case we notice that

$$\begin{aligned} (2i)a + 2 + 2(6k - 3i - 3) &= (2i)a - 6 + 2(6k + 1) - 6i \\ &= (2i+2)a - 6(i+1). \end{aligned}$$

In the latter case, if  $x = (2i)a + 6 + 2(6k - 3i - 3)$ , then

$$x = (2+2i)a - 6i.$$

In the remaining cases we have that

$$x = (2i)a + 8 + 2(6k - 3i - 3) + 2h,$$

namely

$$\begin{aligned} x &= (2i+2)a + 6 - 6(i+1) + 2h \\ &= (2i+2)a - 6i + 2h \end{aligned}$$

for some  $h \geq 0$ . Hence the assertion can be proved as in Case 1.

- *Case 4:*  $x \in [(4k-1)a + 5, \infty[$ .  
 By the characterization of the elements of  $D_{2k-1,k}$  we have that

$$[(4k-1)a + 5, (4k)a] \subseteq G$$

and

$$[(4k)a + 1, (4k)a + 3] = [(4k-1)a + 1, (4k-1)a + 3] + \{a\} \subseteq G.$$

Since

$$4ka + 4 = (4k-2)a + 6 + 12k$$

and  $(4k-2)a + 6 \in C_{2k-1,k}$ , while  $12k \in A_{0,k}$ , we conclude that  $\text{Ap}(G, a) \subseteq [a, 4ka + 4]$ . Hence we conclude that  $[(4k-1)a + 5, \infty[ \subseteq G$ .



- Let  $x \in G$ . Then

$$x = (p + 2q)a + 4r - 6q,$$

where  $\{p, q, r\} \subseteq \mathbb{N}$  and  $0 \leq r \leq q$ .

Let  $j := p + 2q$ .

First we consider the case  $j \geq 4k + 2$ .

Since  $q \leq \frac{j}{2}$ , we have that

$$\begin{aligned} ja + 4r - 6q &\geq j(a - 3) \geq (4k + 2)(a - 3) \\ &= (4k + 2)a - 2(6k + 3) = (4k + 2)a - 2a - 4 \\ &= (4k - 1)a + a - 4 \\ &\geq (4k - 1)a + 9 \end{aligned}$$

because  $a \geq 13$ . Hence  $x \in G$ .

Now we consider the case  $j \leq 4k + 1$ . We notice that

$$ja - 3j = j(a - 3) \geq (4k)(a - 3) > (4k - 2)a.$$

We discuss separately some subcases.

- *Subcase 1:*  $j = 2i + 2$  for some  $i \in [0, k - 2]$ .

We have that  $x \in A_{i,k} \cup B_{i,k}$  because

$$x = (2i + 2)a + 4r - 6q$$

with  $q \leq i + 1$ , namely

$$\begin{aligned} x &\in \{ja - 6(i + 1), ja - 6(i + 1) + 4\} \cup \{ja\} \\ &\quad \text{or} \\ x &= ja - 6i + 2h \end{aligned}$$

for some  $h \in [0, 3i - 1]$ .

- *Subcase 2:*  $j = 2i + 3$  for some  $i \in [0, k - 2]$ .

We can prove as above that  $x \in B_{i,k} \cup A_{i+1,k}$ .

- *Subcase 3:*  $j \in \{2k, 2k + 1\}$ .

We can prove as above that  $x \in A_{k-1,k} \cup B_{k-1,k}$ .

- *Subcase 4:*  $j = 2i + 2$  for some  $i \in [k + 1, 2k - 1]$ .

If  $q \leq k$ , then we can prove as above that  $x \in D_{i,k}$ .

If  $q > k$ , then  $x \in C_{i,k}$ . In fact,  $(2i + 1)a > x$  and

$$\begin{aligned} x &= (2i + 2)a + 4r - 6q \\ &\geq (2i + 2)a - 6q \geq (2i)a + 2(6k + 1) - 6(i + 1) \\ &= (2i)a + 2 + 2(6k - 3i - 3). \end{aligned}$$

In particular,

$$x = (2i)a + 2 + 2(6k - 3i - 3)$$

if  $r = 0$  and  $q = i + 1$ , while

$$x \geq (2i)a + 6 + 2(6k - 3i - 3)$$

in the other cases.

- *Subcase 5:*  $j = 2i + 1$  for some  $i \in [k + 1, 2k]$ .

We can prove as above that  $x \in C_{i,k} \cup D_{i-1,k}$ .

(4) All inequalities follow from the definitions of the sets.

(5) First we notice that

$$|A_{i,k}| = |B_{i,k}| = |C_{i,k}| = |D_{i,k}| = 3i + 3$$

for any suitable  $i$ , while

$$\begin{aligned} |A_{k-1,k}| &= 3k, \\ |B_{k-1,k}| &= |D_{k,k}| = 3k + 3. \end{aligned}$$

We notice that 3 divides the cardinality of all the sets  $A_{i,k}, B_{i,k}, C_{i,k}, D_{i,k}$  and the elements of  $g$  in such sets are obtained (modulo 3) via a concatenation of 3-permutations. We check that this latter assertion is true for the sets  $A_{i,k}$ . The assertion can be verified in a similar way for the remaining sets.

For a fixed index  $i$  we have that

$$\begin{aligned} (2i + 1)a &\equiv 2i + 1 \pmod{3}, \\ (2i + 2)a - 6(i + 1) &\equiv 2i + 2 \pmod{3}, \\ (2i + 2)a - 6(i + 1) + 4 &\equiv 2i \pmod{3}, \end{aligned}$$

while the elements in

$$[(2i + 2)a - 6i, (2i + 2)a - 2]_2$$

are given (modulo 3) by repeated concatenations of the sequence

$$(\overline{2i + 2}, \overline{2i + 2 + 2}, \overline{2i + 2 + 1}).$$

□

**Lemma 4.6.** *Let  $k$  be positive integer and  $a := 3k + 1$ .*

*Let  $S := \{a, 2a - 3, 2a - 2\}$  and  $H_{6,k} := \langle S \rangle$ .*

*Then  $H_{6,k}$  is a 3-permutation semigroup.*

*Proof.* See Lemma 5.2.

□

**Lemma 4.7.** *Let  $k$  be a positive integer and  $a := 3k + 2$ .*

*Let  $S := \{a, a + 1, a + 2\}$ ,  $t := \lfloor \frac{3k}{2} \rfloor$  and*

$$H_{7,k} := \cup_{i=0}^t I_{i,k} \cup [(t + 1)a, \infty[,$$

*where*

$$I_{i,k} := [ia, ia + 2i]$$

*for any  $i \in [0, t + 1]$ .*

*Then the following hold.*

- (1)  $G = H_{7,k}$ .
- (2)  $I_{i,k} < I_{i+1,k}$  for any  $i \in [0, t]$  and  $I_{t,k} < [(t + 1)a, \infty[$ .
- (3)  $H_{7,k}$  is a 3-permutation semigroup.

*Proof.* (1) Let  $x \in I_{i,k}$  for some  $i \in [0, t]$ . Then  $x \in G$  according to Lemma 2.2.

As claimed in the same lemma, we have that

$$\begin{aligned} F(G) &= (3k + 2) \left\lfloor \frac{3k}{2} \right\rfloor + (3k + 1) \\ &= (3k + 2)t + (3k + 1) < (t + 1)a. \end{aligned}$$

Therefore  $[(t + 1)a, \infty[ \subseteq G$ .

Let  $x \in G$ .

If  $x < (t+1)a$ , then  $x \in I_{i,k}$  for some  $i$ , according to Lemma 2.2.

If  $x \geq (t+1)a$ , then  $x \in [(t+1)a, \infty[$ .

Hence  $G \subseteq H_{7,k}$ .

(2) All inequalities can be easily verified.

(3) We notice that  $H_{7,k} \setminus \{0\} = (\cup_{i=1}^t I_{i,k}) \cup [(t+1)a, \infty[$ .

If  $i_1, i_2$  and  $i_3$  are three consecutive positive integers such that  $i_j \equiv j \pmod{3}$  for any  $j \in \{1, 2, 3\}$ , then

$$|I_{i_1,k}| \equiv 0 \pmod{3},$$

$$|I_{i_2,k}| \equiv 2 \pmod{3},$$

$$|I_{i_3,k}| \equiv 1 \pmod{3}.$$

Since 3 divides  $|I_{i_1,k}|$ , we concentrate on the elements of  $I_{i_2}$  and  $I_{i_3}$ .

Let  $x_{i_2}$  and  $y_{i_2}$  be the greatest elements of  $I_{i_2,k}$  and  $z_{i_2}$  the smallest element of  $I_{i_3,k}$ . Then

$$(\overline{x_{i_2}}, \overline{y_{i_2}}, \overline{z_{i_2}}) = (\overline{1}, \overline{2}, \overline{0}).$$

The remaining elements of  $I_{i_1}, I_{i_2}$  and  $I_{i_3}$  can be obtained (modulo 3) through concatenations of 3-permutations. Therefore we conclude that  $H_{7,k}$  is a 3-permutation semigroup.  $\square$

**Lemma 4.8.** *Let  $k$  be a positive integer and  $a := 12k + 2$ .*

*Let  $S := \{a, a+2, a+\frac{a}{2}\}$  and*

$$H_{8,k} := A_{0,k} \cup (\cup_{i=1}^{3k} (A_{i,k} \cup B_{i,k})) \cup (\cup_{i=3k+1}^{6k+1} C_{i,k}) \cup (\cup_{i=3k+1}^{6k} D_{i,k}) \cup E,$$

*where  $A_{0,k} := \{0\}$ , while*

$$A_{i,k} := [ia, ia+2i]_2,$$

$$B_{i,k} := A_{i-1,k} + \left\{a + \frac{a}{2}\right\},$$

*for any  $i \in [1, 3k]$ ,*

$$D_{i,k} := \left[ia + \frac{a}{2}, ia + \frac{a}{2} + 2(i-3k)\right] \cup \left[ia + \frac{a}{2} + 2(i-3k+1), ia + \frac{a}{2} + 6k\right]_2$$

*for any  $i \in [3k+1, 6k]$ ,*

$$C_{3k+1,k} := [(3k+1)a, (3k+1)a+6k]_2,$$

$$C_{3k+2,k} := [(3k+2)a, (3k+2)a+6k]_2,$$

*while*

$$C_{i,k} := D_{i-2,k} + \left\{a + \frac{a}{2}\right\}$$

*for any  $i \in [3k+1, 6k+1]$  and*

$$E := \left[(6k+1)a + \frac{a}{2}, \infty\right[.$$

*Then the following hold.*

(1)  $x \in G$  if and only if

$$x = qa + 2r + s \left(a + \frac{a}{2}\right),$$

*where  $\{q, r, s\} \subseteq \mathbb{N}$  and  $0 \leq r \leq q$ .*

(2)  $G = H_{8,k}$ .

(3) The following inequalities hold:

$$\begin{aligned} A_{i,k} &< B_{i,k} && \text{for any } i \in [1, 3k]; \\ B_{i,k} &< A_{i+1,k} && \text{for any } i \in [1, 3k-1]; \\ B_{3k,k} &< C_{3k+1,k}; \\ C_{i,k} &< D_{i,k} && \text{for any } i \in [3k+1, 6k]; \\ D_{i,k} &< C_{i+1,k} && \text{for any } i \in [3k+1, 6k]; \\ C_{6k+1,k} &< E. \end{aligned}$$

(4)  $H_{8,k}$  is a 3-permutation semigroup.

*Proof.* (1) By definition,  $x \in G$  if and only if

$$x = ta + u(a+2) + v\left(a + \frac{a}{2}\right)$$

for some  $\{t, u, v\} \subseteq \mathbb{N}$ . Since  $ta + u(a+2) \in \langle a, a+2 \rangle$ , according to Lemma 2.2 we can say that  $x \in G$  if and only if

$$x = qa + 2r + v\left(a + \frac{a}{2}\right),$$

where  $\{q, r\} \subseteq \mathbb{N}$  and  $0 \leq r \leq q$ .

(2) Let  $x \in H_{8,k}$ .

If  $x$  belongs to one of the sets  $A_{i,k}, B_{i,k}, C_{i,k}$  or  $D_{i,k}$  for some index  $i$ , then  $x \in G$  in virtue of the way such sets have been defined and the characterization of the elements of  $G$ .

Moreover, we notice that

$$D_{6k,k} \cup C_{6k+1,k} = \left[6ka + \frac{a}{2}, (6k+1)a + 6k\right] \setminus \{(6k+1)a + 6k - 1\}$$

and

$$(6k+2)a + 6k - 1 = (6k+1)a + 6k - 3 + (a+2) \subseteq C_{6k+1,k} + S.$$

Hence  $F(G) = (6k+1)a + 6k - 1$  and  $E \subseteq G$ .

Vice versa, if  $x \in G$  and  $x < (6k+1)a + \frac{a}{2}$ , then

$$x = (s+q)a + 2r + s \cdot \frac{a}{2},$$

where  $s = 2t + \varepsilon$  with  $\varepsilon \in [0, 1]$ . Therefore we can write that

$$x = ia + 2r + \varepsilon \cdot \frac{a}{2},$$

where  $i := t + q$ .

If  $\varepsilon = 0$ , then  $x \in A_{i,k} \cup C_{i,k}$ , while  $x \in B_{i,k} \cup D_{i,k}$  if  $\varepsilon = 1$ .

(3) All inequalities can be easily verified.

(4) If  $i_1, i_2$  and  $i_3$  are three consecutive positive integers such that  $i_j \equiv j \pmod{3}$  for any  $j \in \{1, 2, 3\}$ , then

$$|\cup_{j=1}^3 (A_{i_j} \cup B_{i_j})| \equiv 0 \pmod{3}.$$

For any  $j \in \{1, 2, 3\}$  we have that

$$\begin{aligned} |A_{i_j}| &= i_j + 1 \equiv j + 1 \pmod{3}, \\ |B_{i_j}| &= i_j \equiv j \pmod{3}. \end{aligned}$$

If  $j = 1$ , then  $|A_{i_1} \cup B_{i_1}| \equiv 0 \pmod{3}$ . Let  $x_{i_1}$  and  $y_{i_1}$  be the greatest elements of  $A_{i_1}$  and  $z_{i_1}$  the smallest element of  $B_{i_1}$ . Then

$$(\overline{x_{i_1}}, \overline{y_{i_1}}, \overline{z_{i_1}}) = (\overline{2}, \overline{1}, \overline{0}).$$

We notice in passing that the other elements of  $\overline{g \cap A_{i_1}}$  and  $\overline{g \cap B_{i_1}}$  are obtained through concatenations of 3-permutations. Using a similar argument we can prove that the elements of  $\overline{g \cap (\cup_{j=2}^3 (A_{i_j} \cup B_{i_j}))}$  are obtained through a concatenation of 3-permutations.

As regards the sets  $C_{i,k}$  and  $D_{i,k}$ , first we notice that

$$\begin{aligned} |C_{3k+1,k}| &= 3k+1, \\ |D_{3k+1,k}| &= 3k+2. \end{aligned}$$

The greatest element of  $C_{3k+1,k}$  and the two smallest elements of  $D_{3k+1,k}$  form (modulo 3) a 3-permutation, while the remaining elements of  $\overline{g \cap C_{3k+1,k}}$  and  $\overline{g \cap D_{3k+1,k}}$  are given by concatenations of 3-permutations.

In general we notice that

$$\begin{aligned} |C_{i,k}| &= i-1, \\ |D_{i,k}| &= i+1. \end{aligned}$$

Hence, if we take three consecutive positive integers  $i_1, i_2$  and  $i_3$  such that  $i_j \equiv j \pmod{3}$  for any  $j$ , then we can check as above that the sequence of elements  $\overline{g \cap (\cup_{j=1}^3 (C_{i_j,k} \cup D_{i_j,k}))}$  is given by a concatenation of 3-permutations.  $\square$

**Lemma 4.9.** *Let  $k$  be a positive integer,  $a := 3k+2$ ,  $b := 2a-3$  and  $c := 2a-1$ . Let  $S := \{a, b, c\}$  and*

$$H_{9,k} := (\cup_{i=0}^{2k+1} I_{i,k}) \cup [2ka+4, \infty[,$$

where

$$I_{i,k} := \left\{ ia - 3 \left\lfloor \frac{i}{2} \right\rfloor \right\} \cup \left[ ia - 3 \left\lfloor \frac{i}{2} \right\rfloor + 2, ia \right]$$

for any  $i \in [0, 2k+1]$ .

Then the following hold.

(1)  $x \in G$  if and only if

$$x = (s+2q)a - 3q + 2r,$$

where  $\{q, r, s\} \subseteq \mathbb{N}$  with  $0 \leq r \leq q$ .

(2) We have that

$$I_{i,k} < I_{i+1,k} \quad \text{for any } i \in [0, 2k].$$

(3)  $G = H_{9,k}$ .

(4)  $H_{9,k}$  is a 3-permutation semigroup.

*Proof.* (1) The assertion follows from Lemma 2.2.

(2) The inequality follows from the definition of the sets  $I_{i,k}$ .

(3) Let  $x \in I_{i,k}$  for some  $i$ .

If  $h$  is an integer such that  $0 \leq h \leq 3 \left\lfloor \frac{i}{2} \right\rfloor - 2$ , then we can write

$$-3 \left\lfloor \frac{i}{2} \right\rfloor + 2 + h = -3q + 2r$$

for some  $\{q, r\} \subseteq \mathbb{N}$  such that

$$0 \leq q \leq \left\lfloor \frac{i}{2} \right\rfloor \quad \text{and} \quad 0 \leq r \leq 1.$$

Therefore  $x = (s + 2q)a - 3q + 2r$ , where  $s := i - 2q$ .

Now we notice that

$$\begin{aligned} I_{2k,k} &= \{(2k-1)a + 2\} \cup [(2k-1)a + 4, 2ka], \\ I_{2k+1,k} &= \{2ka + 2\} \cup [2ka + 4, (2k+1)a], \end{aligned}$$

and

$$\begin{aligned} (2k+1)a + 1 &= (2k-1)a + 2 + c, \\ (2k+1)a + 2 &= (2k-1)a + 5 + b, \\ (2k+1)a + 3 &= (2k-1)a + 4 + c. \end{aligned}$$

Therefore  $\text{Ap}(G, a) \subseteq [a, (2k+1)a + 3]$  and consequently  $[2ka + 4, \infty[ \subseteq G$ .

If we take  $x \in G$ , namely

$$x = (s + 2q)a - 3q + 2r,$$

where  $\{q, r, s\} \subseteq \mathbb{N}$  with  $0 \leq r \leq q$ , then  $x \in I_{i,k}$ , where  $i := s + 2q$ . In fact, if  $s = 0$ , then  $q = \frac{i}{2}$ , while  $q \leq \lfloor \frac{i}{2} \rfloor$  if  $s > 0$ . Moreover, in any case we have that

$$-3q + 2r \in \left\{ -3 \left\lfloor \frac{i}{2} \right\rfloor \right\} \cup \left[ -3 \left\lfloor \frac{i}{2} \right\rfloor + 2, 0 \right].$$

Therefore  $G \subseteq H_{9,k}$ .

(4) For any  $i \in [1, 2k]$  we have that

$$|I_{i,k}| \equiv \begin{cases} 1 \pmod{3} & \text{if } i = 1; \\ 0 \pmod{3} & \text{if } i \neq 1. \end{cases}$$

The sequence formed by the greatest element of  $I_{i,k}$  and the two smallest elements of  $I_{i+1,k}$  reads as follows (modulo 3):

$$(\overline{ia}, \overline{(i+1)a}, \overline{(i+1)a + 2}) = (\overline{2i}, \overline{2i+2}, \overline{2i+1}).$$

Moreover the sequence of the remaining elements in  $\overline{g \cap I_{i,k}}$  is given by a concatenation of 3-permutations for any  $i$ .

Finally we notice that the two smallest elements of  $I_{2k+1,k}$  (see above) are  $2ka + 2$  and  $2ka + 4$  and the elements in  $[2ka + 5, \infty[$  can be obtained (modulo 3) through infinitely many concatenations of 3-permutations. Therefore  $H_{9,k}$  is a 3-permutation semigroup.  $\square$

**Lemma 4.10.** *Let  $k$  be a positive integer,  $a := 6k + 3$  and  $t := k - 1$ .*

*Let  $S := \{a, a + 2, 2a - 2\}$  and*

$$H_{10,k} := \{0\} \cup (\cup_{i=0}^t (A_{i,k} \cup B_{i,k})) \cup (\cup_{i=t+1}^{2k} (C_{i,k} \cup D_{i,k})) \cup E,$$

*where*

$$\begin{aligned} A_{i,k} &:= [(1+2i)a - 2i, (1+2i)a - 2i + (6i+2)]_2, \\ B_{i,k} &:= [(2+2i)a - 2(i+1), (2+2i)a - 2(i+1) + (6i+6)]_2, \end{aligned}$$

for any  $i \in [0, t]$ ,

$$\begin{aligned} C_{i,k} &:= [(1+2i)a - 2i, (1+2i)a - 2i + 6(i-t-1) - 2] \\ &\quad \cup [(1+2i)a - 2i + 6(i-t-1), (1+2i)a - 2i + 6k]_2, \\ D_{i,k} &:= [(2+2i)a - 2(i+1), (2+2i)a - 2(i+1) + 6(i-t-1) + 2] \\ &\quad \cup [(2+2i)a - 2(i+1) + 6(i-t-1) + 4, (2+2i)a - 2(i+1) + (6k+2)]_2, \end{aligned}$$

for any  $i \in [t+1, 2k]$ , while

$$E := [(2+4k)a - 2(2k+1), \infty[.$$

Then the following hold.

(1)  $x \in G$  if and only if

$$x = (q+2u)a + 2r - 2u,$$

where  $\{q, r, u\} \subseteq \mathbb{N}$  and  $0 \leq r \leq q$ .

(2)  $G = H_{10,k}$ .

(3) The following inequalities hold:

$$\begin{aligned} A_{i,k} &< B_{i,k} && \text{for any } i \in [0, t], \\ B_{i,k} &< A_{i+1,k} && \text{for any } i \in [0, t-1], \\ B_{t,k} &< C_{t+1,k}, \\ C_{i,k} &< D_{i,k} && \text{for any } i \in [t+1, 2k], \\ D_{i,k} &< C_{i+1,k} && \text{for any } i \in [t+1, 2k-1], \\ D_{2k,k} &< E. \end{aligned}$$

(4)  $H_{10,k}$  is a 3-permutation semigroup.

*Proof.* (1) We have that  $x \in G$  if and only if  $x = sa + t(a+2) + u(2a-2)$  for some  $\{s, t, u\} \subseteq \mathbb{N}$ .

According to Lemma 2.2 we can write

$$x = qa + 2r + u(2a-2) = (q+2u)a + 2r - 2u$$

for some  $\{q, r\} \subseteq \mathbb{N}$  with  $0 \leq r \leq q$ .

(2) Let  $x \in A_{i,k}$  for some  $i$ . Then

$$x = (1+2i)a - 2i + 2h$$

for some  $h \in [0, 3i+1]$ .

If  $x < (1+2i)a$ , then  $-2i + 2h < 0$ . In such a case we can write

$$-2i + 2h = -2\tilde{i} + 2\tilde{h},$$

where  $\tilde{i} \in [1, i]$  and  $\tilde{h} \in [0, 1]$ . Therefore

$$x = (1+2i)a - 2\tilde{i} + 2\tilde{h} = (1+2(i-\tilde{i})+2\tilde{i})a + 2\tilde{h} - 2\tilde{i},$$

namely  $x \in G$ .

If  $x \geq (1+2i)a$ , then  $x \in G$ , since

$$x = (q+2u)a + 2r - 2u,$$

where

$$\begin{aligned} q &:= 1 + 2i, \\ u &:= 0, \\ r &:= h - i \leq 3i + 1 - i = q. \end{aligned}$$

If  $x \in B_{i,k}, C_{i,k}$  or  $D_{i,k}$ , then we can show that  $x \in G$  using an argument as above.

Now we notice that  $D_{k,k}$  contains all integers in the interval

$$[(2 + 4k)a - 2(2k + 1), (2 + 4k)a + 2k].$$

Since in such an interval there are  $6k + 3$  integers, we can say that  $\text{Ap}(G, a) \subseteq [a, (2 + 4k)a + 2k]$ . Therefore  $E \subseteq G$ .

Vice versa, let  $x \in G$ . Then

$$x = (q + 2u)a + 2r - 2u,$$

where  $\{q, r, u\} \subseteq \mathbb{N}$  with  $0 \leq r \leq q$ .

If  $q$  is odd, then  $q = 1 + 2\tilde{q}$  for some  $\tilde{q} \in \mathbb{N}$  and

$$\begin{aligned} x &= (1 + 2(u + \tilde{q}))a + 2r - 2u \\ &= (1 + 2(u + \tilde{q}))a - 2(u + \tilde{q}) + 2(r + \tilde{q}). \end{aligned}$$

Before dealing with some different cases, we notice that

$$r + \tilde{q} \leq q + \tilde{q} = 1 + 3\tilde{q} \leq 1 + 3(u + \tilde{q}).$$

Let  $i := u + \tilde{q}$ .

- If  $0 \leq i \leq t$ , then  $x \in A_{i,k}$ .
- If  $t + 1 \leq i \leq 2k$  and  $i \leq k$ , then  $x \in C_{i,k}$ .
- If  $t + 1 \leq i \leq 2k$  and  $i > k$ , then

$$\begin{aligned} x &\leq (1 + 2i)a - 2i + 2(1 + 3i) \\ &= (2 + 2i)a - 2(i + 1) - a + 2(2 + 3i) \end{aligned}$$

with

$$-a + 2(2 + 3i) \leq 6(i - t - 1) + 2.$$

Therefore  $x \in D_{i,k}$ .

- If  $i \geq 2k + 1$ , then

$$\begin{aligned} x &\geq (1 + 2i)a - 2i = 2i(a - 1) + a \\ &\geq (4k + 2)(a - 1) + a \geq (4k + 2)(a - 2) \\ &= (2 + 4k)a - 2(2k + 1). \end{aligned}$$

Therefore  $x \in E$ .

The proof for  $q$  even can be done in a similar way to the odd case.

(3) All inequalities follow from the definition of the sets.

(4) For any  $i \in [0, t]$  we have that

$$\begin{aligned} |A_{i,k}| &= 3i + 2, \\ |B_{i,k}| &= 3i + 4, \\ |A_{i,k} \cup B_{i,k}| &\equiv 0 \pmod{3}. \end{aligned}$$



The sequence formed by the greatest two elements of  $A_{i,k}$  and the smallest element of  $B_{i,k}$  reads as follows (modulo 3):

$$(\overline{-2i + 2(3i)}, \overline{-2i + 2(3i + 1)}, \overline{-2(i + 1)}) = (\overline{-2i}, \overline{-2i + 2}, \overline{-2i + 1}).$$

This latter is a 3-permutation and the sequences of the remaining elements of  $g \cap A_{i,k}$  and  $g \cap B_{i,k}$  are obtained through concatenations of 3-permutations.

As regards the sets  $C_{i,k}$  and  $D_{i,k}$ , we have that

$$\begin{aligned} |C_{t+1,k}| &= 3k + 1, \\ |D_{t+1,k}| &= 3k + 3, \end{aligned}$$

while

$$\begin{aligned} |C_{i,k}| &= 3(i - t - 1) + 3k, \\ |D_{i,k}| &= 3(i - t - 1) + 3k + 3, \end{aligned}$$

for any  $i \in [t + 2, 2k]$ .

If  $i \in [t + 1, 2k]$ , then the sequence formed by the greatest element of  $C_{i,k}$  and the two smallest elements of  $D_{i,k}$  reads as follows (modulo 3):

$$(\overline{-2i}, \overline{-2i + 1}, \overline{-2i + 2}).$$

If  $i \in [t + 1, 2k - 1]$ , then the sequence formed by the greatest element of  $D_{i,k}$  and the two smallest elements of  $C_{i+1,k}$  reads as follows (modulo 3):

$$(\overline{-2i}, \overline{-2i + 1}, \overline{-2i + 2}).$$

Finally, the sequence formed by the greatest element of  $D_{2k,k}$  and the two smallest elements of  $E$  reads as follows (modulo 3):

$$(\overline{2k}, \overline{2k + 1}, \overline{2k + 2}).$$

Therefore  $H_{10,k}$  is a 3-permutation semigroup. □

**Lemma 4.11.** *Let  $k$  be a positive integer,  $a := 6k + 4$  and  $t := \frac{a-2}{2}$ .*

*Let  $S := \{a, b, c\}$ , where  $b := a + 1$  and  $c := a + \frac{a}{2}$ .*

*Let*

$$H_{11,k} := (\cup_{i=0}^t (A_{i,k} \cup B_{i,k})) \cup C,$$

*where  $A_{0,k} := \{0\}$ ,  $B_{0,k} := \emptyset$ , while*

$$\begin{aligned} A_{i,k} &:= [ia, ia + i], \\ B_{i,k} &:= A_{i-1,k} + \{c\}, \end{aligned}$$

*for any  $i \in [1, t]$ , and*

$$C := [(t + 1)a, \infty[.$$

*Then the following hold.*

(1)  *$x \in G$  if and only if*

$$x = (q + u)a + u \cdot \frac{a}{2} + r,$$

*where  $\{q, r, u\} \subseteq \mathbb{N}$  with  $0 \leq r \leq q$ .*

(2)  *$G = H_{11,k}$ .*

(3) The following inequalities hold:

$$\begin{aligned} A_{i,k} &< B_{i,k} && \text{for any } i \in [0, t], \\ B_{i,k} &< A_{i+1,k} && \text{for any } i \in [0, t-1], \\ B_{t,k} &< C. \end{aligned}$$

(4)  $H_{11,k}$  is a 3-permutation semigroup.

*Proof.* (1) We have that  $x \in G$  if and only if

$$x = sa + tb + uc$$

for some  $\{s, t, u\} \subseteq \mathbb{N}$ . According to Lemma 2.2 we can write

$$x = qa + r + u \left( a + \frac{a}{2} \right) = (q + u)a + u \cdot \frac{a}{2} + r$$

for some  $\{q, r, u\} \subseteq \mathbb{N}$  with  $r \in [0, q]$ .

(2) If  $x \in A_{i,k}$  for some  $i \in [0, t]$ , then  $x \in \langle a, b \rangle \subseteq G$  according to Lemma 2.2.

If  $x \in B_{i,k}$  for some  $i \in [0, t]$ , then  $x \in G$  by the definition of the set  $B_{i,k}$ .

Now we notice that

$$\begin{aligned} A_{t,k} &= \left[ ta, ta + \frac{a}{2} - 1 \right], \\ B_{t,k} &= \left[ ta + \frac{a}{2}, (t+1)a - 2 \right]. \end{aligned}$$

Therefore

$$[(t+1)a, (t+2)a - 1] = (A_{t,k} \cup B_{t,k}) + \{a, b\} \subseteq G,$$

namely  $\text{Ap}(G, a) \subseteq [a, (t+2)a - 1]$  and consequently  $C \subseteq G$ .

Vice versa, let  $x \in G$ .

If  $x \geq (t+1)a$ , then  $x \in C$ .

If  $x < (t+1)a$ , then we can write

$$x = (q + u)a + u \cdot \frac{a}{2} + r,$$

where  $\{q, r, u\} \subseteq \mathbb{N}$  and  $0 \leq r \leq q$ .

If  $u$  is odd, then  $u = 2v + 1$  for some  $v \in \mathbb{N}$ . Therefore

$$x = (q + 2v + 1)a + (2v + 1)\frac{a}{2} + r = (q + 3v + 1)a + \frac{a}{2} + r.$$

Hence  $x \in B_{q+3v,k}$ .

If  $u$  is even, then we can show as above that  $x \in A_{q+3v,k}$ , where  $v := \frac{u}{2}$ .

(3) All inequalities follow immediately by the definition of the sets.

(4) For any  $i \in [0, t]$  we have that

$$\begin{aligned} |A_{i,k}| &= i + 1, \\ |B_{i,k}| &= i. \end{aligned}$$

Let  $i_1, i_2$  and  $i_3$  be three consecutive positive integers with  $i_j \equiv j \pmod{3}$  for any  $j \in \{1, 2, 3\}$ . Then

$$|\cup_{j=1}^3 (A_{i_j} \cup B_{i_j})| \equiv 0 \pmod{3}.$$

The sequence formed by the greatest two elements of  $A_{i_1,k}$  and the smallest element of  $B_{i_1,k}$  reads as follows (modulo 3):

$$(\overline{i_1 a + i_1 - 1}, \overline{i_1 a + i_1}, \overline{(i_1 - 1)a}) = (\overline{1}, \overline{2}, \overline{0}).$$

The sequences of the other elements of  $\overline{g \cap A_{i_1,k}}$  and  $\overline{g \cap B_{i_1,k}}$  are obtained through concatenations of 3-permutations.

The sequence formed by the greatest two elements of  $B_{i_2,k}$  and the smallest element of  $A_{i_3,k}$  reads as follows (modulo 3):

$$(\overline{i_1 a + i_1 - 1 + c}, \overline{i_1 a + i_1 + c}, \overline{i_3 a}) = (\overline{1}, \overline{2}, \overline{0}).$$

The sequences of the remaining elements of  $g$  belonging  $A_{i_2,k}$ ,  $B_{i_2,k}$ ,  $A_{i_3,k}$  and  $B_{i_3,k}$  are obtained (modulo 3) through concatenations of 3-permutations.

Finally we notice that  $t \equiv 1 \pmod{3}$ . As proved above, we have that  $\overline{g \cap (A_{t,k} \cup B_{t,k})}$  is given by a concatenation of 3-permutations.  $\square$

**Lemma 4.12.** *Let  $k$  be a positive integer and  $a := 12k + 4$ .*

*Let  $S := \{a, b, c\}$ , where  $b := a + \frac{a}{2} - 3$  and  $c := a + \frac{a}{2} - 1$ .*

*Let*

$$H_{12,k} := (\cup_{i=0}^{3k} (A_{i,k} \cup B_{i,k})) \cup (\cup_{j=1}^{3k+1} (C_{j,k} \cup D_{j,k})) \cup E,$$

*where  $A_{0,k} := \{0\}$ ,  $B_{0,k} := \emptyset$ , and*

$$A_{i,k} := \left[ ia - 6 \left\lfloor \frac{i}{3} \right\rfloor, ia \right]_2,$$

$$B_{i,k} := \left[ ia + \frac{a}{2} - 3 - 6 \left\lfloor \frac{i-1}{3} \right\rfloor, ia + \frac{a}{2} - 1 \right]_2,$$

*for any  $i \in [1, 3k]$ , while*

$$C_{1,k} := [(3k+1)a - 6k, (3k+1)a - 4]_2 \cup [(3k+1)a - 2, (3k+1)a],$$

$$C_{2,k} := [(3k+2)a - 6k, (3k+2)a - 4]_2 \cup [(3k+2)a - 2, (3k+2)a],$$

*and*

$$C_{j,k} := \left[ (3k+j)a - (6k+2), (3k+j)a - 4 - 6 \left\lfloor \frac{j-1}{3} \right\rfloor \right]_2$$

$$\cup \left[ (3k+j)a - 2 - 6 \left\lfloor \frac{j-1}{3} \right\rfloor, (3k+j)a \right]$$

*for any  $j \in [3, 3k+1]$ , while*

$$D_{j,k} := \left[ (3k+j)a + \frac{a}{2} + 1 - (6k+2), (3k+j)a + \frac{a}{2} - 1 - 6 \left\lfloor \frac{j+1}{3} \right\rfloor \right]_2$$

$$\cup \left[ (3k+j)a + \frac{a}{2} + 1 - 6 \left\lfloor \frac{j+1}{3} \right\rfloor, (3k+j)a + \frac{a}{2} - 1 \right]$$

*for any  $j \in [1, 3k+1]$  and*

$$E := [(6k+1)a + 3, \infty[.$$

*Then the following hold.*

(1)  $x \in G$  if and only if

$$x = (s+q)a + q \cdot \frac{a}{2} - 3q + 2r,$$

*where  $\{s, q, r\} \subseteq \mathbb{N}$  with  $0 \leq r \leq q$ .*

(2)  $G = H_{12,k}$ .

(3) The following inequalities hold:

$$\begin{aligned} A_{i,k} &< B_{i,k} && \text{for any } i \in [1, 3k], \\ B_{i,k} &< A_{i+1,k} && \text{for any } i \in [1, 3k-1], \\ B_{3k,k} &< C_{1,k}, \\ C_{j,k} &< D_{j,k} && \text{for any } j \in [1, 3k+1], \\ D_{j,k} &< C_{j+1,k} && \text{for any } j \in [1, 3k], \\ D_{3k+1,k} &< E. \end{aligned}$$

(4)  $H_{12,k}$  is a 3-permutation semigroup.

*Proof.* (1) We have that  $x \in G$  if and only if

$$\begin{aligned} x &= sa + tb + uc = sa + qb + 2r \\ &= (s+q)a + q \cdot \frac{a}{2} - 3q + 2r \end{aligned}$$

for some  $\{q, r, s\} \subseteq \mathbb{N}$  with  $0 \leq q \leq r$  according to Lemma 2.2.

(2) Let  $x \in H_{12,k}$ .

If  $x \in A_{i,k}$  for some  $i \in [1, 3k]$ , then

$$x = ia - 6 \left\lfloor \frac{i}{3} \right\rfloor + 2h$$

for some  $h \in [0, 3 \lfloor \frac{i}{3} \rfloor]$ .

We can write

$$-6 \left\lfloor \frac{i}{3} \right\rfloor + 2h = -6l + 2r$$

for some integers  $l$  and  $r$  such that  $0 \leq l \leq \lfloor \frac{i}{3} \rfloor$  and  $0 \leq r \leq 2$ . In particular we notice that  $r = 0$  if  $l = 0$ .

If we set

$$\begin{aligned} s &:= i - 3l, \\ q &:= 2l, \end{aligned}$$

then we have that

$$x = (s+q)a + q \cdot \frac{a}{2} - 3q + 2r,$$

namely  $x \in G$ .

As regards the sets  $B_{i,k}$ , we notice that

$$B_{i,k} = A_{i-1,k} + \{b, c\}$$

for any  $i \in [1, 3k]$ .

Now we notice that

$$\begin{aligned} C_{1,k} &= (A_{3k,k} + \{a\}) \cup \left( \left\{ 3ka - 6 \left\lfloor \frac{3k}{3} \right\rfloor \right\} + \{b\} \right), \\ C_{2,k} &= C_{1,k} + \{a\}, \\ C_{3,k} &= (C_{2,k} + \{a\}) \cup \left( \left\{ 3ka - 6 \left\lfloor \frac{3k}{3} \right\rfloor \right\} \right) + \{2c\}. \end{aligned}$$

Now we consider  $x \in C_{j,k}$  for some  $j \in [3, 3k+1]$ . If

$$x = (3k+j)a - (6k+2) + 2h$$

for some  $h \in [0, 3k + 1]$ , then we can write

$$x = (s + q)a + q \cdot \frac{a}{2} - 3q + 2r,$$

where

$$\begin{aligned} s &:= 3k + j - 3l, \\ q &:= 2l, \end{aligned}$$

for suitable values of  $l$  and  $r$  such that

$$-(6k + 2) + 2h = -6l + 2r$$

with  $0 \leq l \leq k + 1$  and  $0 \leq r \leq 2$ .

Therefore, if  $x \in C_{j,k}$  for some  $j \equiv 1 \pmod{3}$  with

$$x = (3k + j)a - (6k + 2) + 2h$$

for some  $h \in [0, 3k + 1]$ , then  $x \in G$ .

Now we take  $x \in C_{j,k}$  for some  $j \equiv 1 \pmod{3}$  with

$$x = (3k + j)a - 1 - 6 \left\lfloor \frac{j-1}{3} \right\rfloor + 2h$$

for some  $h \in [0, j - 1]$ . Then

$$x = (3k + j)a - 1 - 6 \left( \frac{j-1}{3} \right) + 2h = (3k + j)a + 1 - 2j + 2h.$$

If we set

$$\begin{aligned} s &:= 0, \\ q &:= 1 + 2 \left( \frac{3k + j - 1}{3} \right), \\ r &:= h, \end{aligned}$$

then

$$x = (s + q)a + q \cdot \frac{a}{2} - 3q + 2r.$$

In particular we notice that  $0 \leq r \leq q$ . In fact,

$$\begin{aligned} r \leq j - 1 &= 3 \left( \frac{j-1}{3} \right) \leq 2k + \frac{j-1}{3} \\ &\leq 2k + 2 \cdot \frac{j-1}{3} + 1 = q. \end{aligned}$$

Therefore, if  $x \in C_{j,k}$  with  $j \equiv 1 \pmod{3}$ , then  $x \in G$ . In the case that  $j \equiv 2 \pmod{3}$  (resp.  $j \equiv 3 \pmod{3}$ ), then

$$C_{j,k} = C_{j-1,k} + \{a\} \quad (\text{resp. } C_{j-2,k} + \{a\}).$$

As regards the sets  $D_{j,k}$ , we have that

$$D_{1,k} = A_{3k,k} + \{b, c\}$$

and

$$D_{j,k} = C_{j-1,k} + \{b, c\}$$

for any  $j \in [2, 3k + 1]$ .

Now we notice that

$$C_{3k+1,k} = \left[ (6k+1)a - \frac{a}{2}, (6k+1)a \right],$$

$$D_{3k+1,k} = \{(6k+1)a + 1\} \cup \left[ (6k+1)a + 3, (6k+1)a + \frac{a}{2} - 1 \right],$$

namely

$$C_{3k+1,k} \cup D_{3k+1,k} = \left[ (6k+1)a - \frac{a}{2}, (6k+1)a + \frac{a}{2} - 1 \right] \setminus \{(6k+1)a + 2\}.$$

Moreover we have that

$$(6k+2)a + 2 = (6k+1)a + 3 + c.$$

Therefore  $\text{Ap}(G, a) \subseteq [a, (6k+2)a + 2]$  and  $E \subseteq G$ .

Vice versa, let  $x \in G$ .

If  $x \geq (6k+1)a + 3$ , then  $x \in E \subseteq H_{12,k}$ .

If  $x < (6k+1)a + 3$ , then

$$x = (s+q)a + q \cdot \frac{a}{2} - 3q + 2r,$$

where  $\{s, q, r\} \subseteq \mathbb{N}$  with  $0 \leq r \leq q$ .

We deal in detail with the case that  $q$  is even, namely  $q = 2\tilde{q}$  for some  $\tilde{q} \in \mathbb{N}$ .

We can write

$$x = (s + 3\tilde{q})a - 6\tilde{q} + 2r.$$

We set  $i := s + 3\tilde{q}$  and consider some cases.

- *Case 1:*  $i \leq 3k + 2$ .

We notice that  $x \in A_{i,k} \cup C_{1,k} \cup C_{2,k}$  because

$$\tilde{q} \leq \left\lfloor \frac{i}{3} \right\rfloor \leq k.$$

- *Case 2:*  $3k + 3 \leq i \leq 6k + 1$ .

We have that

$$i = 3k + j$$

for some  $j \in [3, 3k + 1]$ .

If  $-6\tilde{q} + 2r \geq -(6k + 2)$ , then  $x \in C_{j,k}$ .

Conversely we have that

$$-6k - 6 \left\lfloor \frac{j}{3} \right\rfloor \leq -6\tilde{q} + 2r \leq -6k - 3.$$

Hence

$$\begin{aligned} x &= ia - a + a - 6\tilde{q} + 2r \\ &\geq (i-1)a + a - 6k - 6 \left\lfloor \frac{j}{3} \right\rfloor \\ &= (i-1)a + a - \frac{a}{2} + 2 - 6 \left\lfloor \frac{j}{3} \right\rfloor \\ &= (3k+j-1)a + \frac{a}{2} + 2 - 6 \left\lfloor \frac{j}{3} \right\rfloor, \end{aligned}$$

namely  $x \in D_{j-1,k}$ .

- *Case 3:  $i \geq 6k + 2$ .*

First we notice that

$$x \geq ia - 6\tilde{q} \geq ia - 6 \left\lfloor \frac{i}{3} \right\rfloor.$$

Moreover the sequence

$$\{\delta_i\}_{i=6k+2}^\infty := \left\{ ia - 6 \left\lfloor \frac{i}{3} \right\rfloor \right\}_{i=6k+2}^\infty$$

is increasing. Indeed, if  $i \in [6k + 2, \infty[$ , we have that

$$\delta_{i+1} - \delta_i = a - 6 \left( \left\lfloor \frac{i+1}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor \right) \geq a - 6 > 0.$$

Hence we conclude that

$$\begin{aligned} x &\geq ia - 6 \left\lfloor \frac{i}{3} \right\rfloor \geq (6k + 2)a - 6 \left\lfloor \frac{6k + 2}{3} \right\rfloor \\ &= (6k + 2)a - 12k = (6k + 1)a + 4, \end{aligned}$$

namely  $x \in E$ .

(3) All inequalities follow from the definition of the sets.

(4) For any  $i \in [1, 3k]$  we have that

$$\begin{aligned} |A_{i,k}| &= 3 \left\lfloor \frac{i}{3} \right\rfloor + 1, \\ |B_{i,k}| &= \left\lfloor \frac{i-1}{3} \right\rfloor + 2. \end{aligned}$$

Therefore  $|A_{i,k} \cup B_{i,k}| \equiv 0 \pmod{3}$ . The sequence formed by the greatest element of  $A_{i,k}$  and the smallest two elements of  $B_{i,k}$  is (modulo 3) a 3-permutation and the sequences of the remaining elements of  $\bar{g} \cap A_{i,k}$  and  $\bar{g} \cap B_{i,k}$  can be obtained through concatenations of 3-permutations. Since the same holds for the elements in  $C_{j,k}$  and  $D_{j,k}$  for any  $j$ , we conclude that  $H_{12,k}$  is a 3-permutation semigroup.  $\square$

**Lemma 4.13.** *Let  $k$  be a positive integer,  $a := 6k + 5$ ,  $b := a + \frac{a-3}{2}$  and  $c := b + 1$ .*

*Let  $S := \{a, b, c\}$  and*

$$H_{13,k} := \{0\} \cup \left( \bigcup_{i=0}^k I_{i,k} \right) \cup C,$$

*where  $I_{i,k} := A_{i,k} \cup B_{i,k}$  and*

$$A_{i,k} := ([a, a + 3i] \cup [b, b + 1 + 3i] \cup [2a, 2a + 3i]) + \{2ib\},$$

$$B_{i,k} := ([a, a + 1 + 3i] \cup [b, b + 3 + 3i] \cup [2a, 2a + 1 + 3i]) + \{(2i + 1)b\},$$

*for any  $i \in [0, k]$ , while*

$$C := [2a + 2kb, \infty[.$$

*Then the following hold.*

- (1)  $H_{13,k}$  is a co-finite submonoid of  $G$  containing  $S$ .
- (2)  $H_{13,k}$  is a 3-permutation semigroup.

*Proof.* We notice that the claim is true when  $k = 1$  by a direct computation (see Table 1 in Section 5).

In the following we suppose that  $k \geq 2$ .

(1) We prove by induction on  $i$  that

$$A_{i,k} \cup B_{i,k} \subseteq G.$$

First we observe that

$$A_{0,k} = \{a\} \cup \{b, c\} \cup \{2a\} = S \cup \{2a\} \subseteq G.$$

Since

$$\begin{aligned} [a + b, a + b + 1] &= \{a\} + \{b, c\}, \\ [2b, 2b + 3] &= ([b, c] + [b, c]) \cup \{3a\}, \\ [2a + b, 2a + b + 1] &= [a + b, a + b + 1] + \{a\}, \end{aligned}$$

we can say that  $B_{0,k} \subseteq G$ .

Now let  $i > 0$ . We have that

$$\begin{aligned} [a, a + 3i] + \{2ib\} &= ([2b, 2b + 3 + 3(i - 1)] + \{2(i - 1)b\}) + \{a\}, \\ [b, b + 1 + 3i] + \{2ib\} &= ([2b, 2b + 3 + 3(i - 1)] + \{2(i - 1)b\}) + \{b\} \\ &\quad \cup (([a + b + 1 + 3(i - 1)] + \{2(i - 1)b\}) + \{2a\}), \\ [2a, 2a + 3i] + \{2ib\} &= ([a, a + 3i] + \{2ib\}) + \{a\}. \end{aligned}$$

Therefore  $A_{i,k} \subseteq B_{i-1,k} + \{a, b, 2a\} \subseteq G$ .

In a similar way we can prove that  $B_{i,k} \subseteq A_{i-1,k} + \{a, b, c\}$ .

As regards the set  $C$ , we notice that

$$\begin{aligned} 2a + 3k &= a + b - 1, \\ a + b + 1 + 3k &= 2b, \\ 2b + 3 &= 3a. \end{aligned}$$

Therefore  $[2a + 2kb, 3a + 2kb] \subseteq G$ , namely  $\text{Ap}(G, a) \subseteq [a, 3a + 2kb]$ . We conclude that  $C \subseteq G$ .

Now we take  $x \in I_{i_1,k}$  and  $y \in I_{i_2,k}$  for some  $\{i, i_2\} \subseteq [0, k]$ . If  $x + y < 2a + 2kb$ , then  $x + y$  belongs to one of the sets in rows 2 – 3, columns 2 – 3 of the table below, where  $i_3 := i_1 + i_2$ .

	$A_{i_2,k}$	$B_{i_2,k}$
$A_{i_1,k}$	$I_{i_3,k}$	$I_{i_3,k} \cup I_{i_3+1,k}$
$B_{i_1,k}$		$I_{i_3+1,k} \cup I_{i_3+2,k}$

(2) The following inequalities hold for any  $i \in [0, k - 1]$ :

- $a + 3i < b$ ;
- $b + 1 + 3i < 2a$ ;
- $2a + 3i < a + b$ ;
- $a + b + 1 + 3i < 2b$ ;
- $2b + 3 + 3i < 2a + b$ ;
- $2a + b + 1 + 3i + 2ib < a + 2(i + 1)b$ .

Moreover we have that

- $a + 3k < b$ ;
- $b + 1 + 3k < 2a$ .

We notice that

$$A_{i,k} < B_{i,k} \quad \text{and} \quad B_{i,k} < A_{i+1,k}$$

and

$$|A_{i,k}| \equiv 1 \pmod{3}, \quad B_{i,k} \equiv 2 \pmod{3}$$



for any  $i \in [0, k-1]$ . Therefore

$$|A_{i,k} \cup B_{i,k}| \equiv 0 \pmod{3}$$

for any  $i \in [0, k-1]$ . From the definition of the sets  $A_{i,k}$  and  $B_{i,k}$  one can verify that the sequence  $\overline{g \cap (A_{i,k} \cup B_{i,k})}$  is given by a concatenation of 3-permutations for any  $i$ . The same holds for the sequence  $\overline{g \cap (A_{k,k} \cup C)}$ . Hence we conclude that  $H_{13,k}$  is a 3-permutation semigroup.  $\square$

**Lemma 4.14.** *Let  $k$  be a positive integer,  $a := 6k+5$ ,  $b := 2a-6$  and  $c := 2a-4$ . Let  $S := \{a, b, c\}$  and*

$$H_{14,k} := \left( \bigcup_{i=0}^{4k+2} I_{i,k} \right) \cup [(4k+1)a+3, \infty[,$$

where

$$I_{i,k} := \left[ ia - 6 \left\lfloor \frac{i}{2} \right\rfloor, ia - 4 \right]_2 \cup \{ia\}$$

for any  $i \in [0, 4k+2]$ .

Then the following hold.

(1)  $x \in G$  if and only if

$$x = (s+2q)a - 6q + 2r,$$

where  $\{q, r, s\} \subseteq \mathbb{N}$  with  $0 \leq r \leq q$ .

(2) We have that

$$\begin{aligned} I_{i,k} &< I_{i+2,k} && \text{if } i \in [0, 4k], \\ I_{i,k} &< I_{i+1,k} && \text{if } i \in [0, 2k], \end{aligned}$$

while for any  $i \in [2k+1, 4k+1]$  we have that

$$I_{i+1,k} \cap [(i-1)a+1, ia] = [ia - 6i_k - 1, ia - 1]_2,$$

where  $i_k := \left\lfloor \frac{i-1-2k}{2} \right\rfloor$ .

(3)  $G = H_{14,k}$ .

(4)  $H_{14,k}$  is a 3-permutation semigroup.

*Proof.* We notice that the claim is true when  $k=1$  by a direct computation (see Table 1 in Section 5).

In the following we suppose that  $k \geq 2$ .

(1) The assertion follows from Lemma 2.2.

(2) All assertions follow from the definition of the sets.

(3) Let  $x \in I_{i,k}$  for some  $i \in [0, 4k+2]$ .

If  $x = ia$ , then  $x \in G = \langle S \rangle$ .

If  $x < ia$ , then

$$x = ia - 6 \left\lfloor \frac{i}{2} \right\rfloor + 2h$$

for some integer  $h$  such that  $0 \leq h \leq 3 \left\lfloor \frac{i}{2} \right\rfloor - 2$ .

We can write

$$-6 \left\lfloor \frac{i}{2} \right\rfloor + 2h = -6q + 2r$$

for some  $\{q, r\} \subseteq \mathbb{N}$  such that

$$0 \leq q \leq \left\lfloor \frac{i}{2} \right\rfloor \quad \text{and} \quad 0 \leq r \leq 2.$$

In particular we notice that  $r = 0$  if  $q = 0$ , while  $r \leq 1$  if  $q = 1$  due to the restrictions on  $h$ . Therefore  $r \leq q$  whichever the value of  $q$  is.

Hence  $x = (s + 2q)a - 6q + 2r$ , where  $s := i - 2q$ .

Now we want to prove that any  $x \geq (4k + 1)a + 3$  belongs to  $G$ .

First we notice that

$$I_{4k+1,k} \cap [4ka + 1, (4k + 1)a] = [4ka + 1, 4ka + 1 + 6k]_2,$$

$$I_{4k+2,k} \cap [4ka + 1, (4k + 1)a] = [4ka + 4, 4ka + 4 + 6k]_2.$$

Therefore

$$\{4ka + 1\} \cup [4ka + 3, (4k + 1)a - 3] \cup \{(4k + 1)a - 1\} \subseteq G.$$

Since

$$[4ka + 4, 4ka + 8] + \{b\} = [(4k + 2)a - 2, (4k + 2)a + 2],$$

we can say that

$$[(4k + 1)a + 3, (4k + 2)a + 2] \subseteq G.$$

Hence  $\text{Ap}(G, a) \subseteq [a, (4k + 2)a + 2]$  and  $[(4k + 1)a + 3, \infty[ \subseteq G$ .

Now we take  $x \in G$ . We have that

$$x = (s + 2q)a - 6q + 2r$$

for some  $\{q, r, s\} \subseteq \mathbb{N}$  with  $0 \leq r \leq q$ .

Let  $i := s + 2q$ . Then  $x \in I_{i,k}$ . In fact,  $q = \frac{i}{2}$  if  $s = 0$ , while  $q \leq \lfloor \frac{i}{2} \rfloor$  if  $s > 0$ , and

$$-6q + 2r = 0 \quad \text{or} \quad -6q + 2r \leq -4.$$

(4) For any  $i \in [1, 4k + 2]$  we define  $U_i := [(i - 1)a + 1, ia] \cap G$ .

If  $i \in [1, 2k]$ , then

$$|U_i| \equiv \begin{cases} 1 \pmod{3} & \text{if } i = 1, \\ 0 \pmod{3} & \text{if } i \neq 1. \end{cases}$$

For any  $i \in [2, 2k]$  the sequence (modulo 3) formed by the greatest element of  $U_{i-1}$  and the two smallest elements of  $U_i$  reads as follows:

$$(\overline{2i - 2}, \overline{2i}, \overline{2i + 2}).$$

The sequence of the remaining elements of  $\overline{g \cap U_i}$  can be obtained through a concatenation of 3-permutations.

If  $i \in [2k + 1, 4k + 1]$ , then

$$U_i = (I_{i,k} \cap [(i - 1)a + 1, ia - 6i_k - 4]) \cup [ia - 6i_k - 2, ia - 3] \cup \{ia - 1, ia\}.$$

In particular, if  $i \in \{2k + 1, 2k + 2\}$ , then

$$U_i = (I_{i,k} \cap [(i - 1)a + 1, ia - 4]) \cup \{ia - 1, ia\}.$$

For  $i = 2k + 1$  we have that

$$|(I_{i,k} \cap [(i - 1)a + 1, ia - 4])| \equiv 2 \pmod{3}.$$

For any  $i \in [2k + 2, 4k + 1]$  we have that

$$|I_{i,k} \cap [(i - 1)a + 1, ia - 6i_k - 4]| \equiv 1 \pmod{3},$$

while

$$|[ia - 6i_k - 2, ia - 3]| \equiv 0 \pmod{3},$$

provided that  $i > 2k + 2$ . Therefore

$$|U_i| \equiv 0 \pmod{3}$$

for any  $i \in [2k + 3, 4k + 1]$ .

Now we observe that the sequence (modulo 3) formed by the greatest element of  $U_{2k}$  and the two smallest elements of  $U_{2k+1}$  reads as follows:

$$(\overline{k}, \overline{k+2}, \overline{k+1}).$$

For any  $i \in [2k + 2, 4k + 2]$  we have that the sequence (modulo 3) formed by the greatest two elements of  $U_{i-1}$  and the smallest element of  $U_i$  reads as follows:

$$(\overline{(i-1)a-1}, \overline{(i-1)a}, \overline{(i-1)a+1}).$$

The sequence of the remaining elements of  $\overline{g \cap U_i}$  is given by a concatenation of 3-permutations for any  $i \in [2k + 1, 4k + 1]$ .

As regards the set  $U_{4k+2}$ , we notice that

$$U_{4k+2} = \{(4k+1)a+1\} \cup [(4k+1)a+3, (4k+2)a].$$

Therefore the sequence of elements greater than  $(4k+1)a+1$  belonging to  $H_{14,k}$  reads (modulo 3) as an infinite concatenation of 3-permutations.  $\square$

**Lemma 4.15.** *Let  $k$  be a positive integer,  $a := 12k + 8$ ,  $b := a + 2$  and  $c := b + \frac{b}{2}$ . Let  $S := \{a, b, c\}$  and*

$$H_{15,k} := (\cup_{i=0}^{6k+4} (A_{i,k} \cup B_{i,k})) \cup C,$$

where  $A_{0,k} := \{0\}$ ,  $B_{0,k} := \emptyset$ , and

$$\begin{aligned} A_{i,k} &:= [ia, ia + 2i]_2, \\ B_{i,k} &:= \left[ ia + \frac{a}{2} + 3, ia + \frac{a}{2} + 3 + 2(i-1) \right]_2, \end{aligned}$$

for any  $i \in [1, 6k + 4]$ , while

$$C := \left[ (6k+4)a + \frac{a}{2} + 3, \infty \right[.$$

If

$$\begin{aligned} U_{i,k} &:= \left[ ia, ia + \frac{a}{2} + 2 \right], \\ V_{i,k} &:= \left[ ia + \frac{a}{2} + 3, (i+1)a - 1 \right], \end{aligned}$$

for any  $i \in [1, 6k + 4]$ , then the following hold.

(1)  $x \in G$  if and only if

$$x = (q + 3v + \varepsilon)a + \varepsilon \cdot \frac{a}{2} + 6v + 3\varepsilon + 2r,$$

where  $\{q, r, v\} \subseteq \mathbb{N}$ ,  $\varepsilon \in \{0, 1\}$  and  $0 \leq r \leq q$ .

(2)  $G = H_{15,k}$ .

(3) The following inequalities hold:

$$\begin{array}{ll}
U_{i,k} < V_{i,k} & \text{for any } i, \\
A_{i,k} < A_{i+1,k} & \text{if } i \in [1, 6k+3], \\
B_{i,k} < B_{i+1,k} & \text{if } i \in [1, 6k+3], \\
V_{i,k} < U_{i+1,k} & \text{if } i \in [1, 6k+3], \\
A_{i,k} < V_{i,k} & \text{if } i \in [1, 3k+3], \\
A_{i,k} \cap V_{i,k} = [ia + \frac{a}{2} + 3, ia + 2i]_2 & \text{if } i \in [3k+4, 6k+4], \\
B_{i,k} < U_{i+1,k} & \text{if } i \in [1, 3k+1], \\
B_{i,k} \cap U_{i+1,k} = [(i+1)a, ia + \frac{a}{2} + 3 + 2(i-1)]_2 & \text{if } i \in [3k+2, 6k+3].
\end{array}$$

(4)  $H_{15,k}$  is a 3-permutation semigroup.

*Proof.* (1) We have that  $x \in G$  if and only if

$$x = sa + t(a+2) + u \left( a + 2 + \frac{a+2}{2} \right)$$

for some  $\{s, t, u\} \subseteq \mathbb{N}$ . According to Lemma 2.2 we can write

$$x = qa + 2r + u \left( a + 2 + \frac{a+2}{2} \right)$$

for some  $\{q, r, u\} \subseteq \mathbb{N}$  such that  $0 \leq r \leq q$ . If we write

$$u = 2v + \varepsilon$$

for some  $v \in \mathbb{N}$  and  $\varepsilon \in \{0, 1\}$ , then the result follows.

(2) If  $x \in A_{i,k}$  for some  $i \geq 1$ , then  $x = ia + 2h$  for some  $h \in [0, i]$ . Therefore  $x = qa + 2r$ , where  $q := i$  and  $r := h$ , namely  $x \in G$ .

If  $x \in B_{i,k}$  for some  $i \geq 1$ , then  $x \in A_{i-1} + \{c\} \subseteq G$ .

Now we notice that

$$\begin{aligned}
A_{6k+3,k} &= [(6k+3)a, (6k+4)a - 2]_2, \\
B_{6k+3,k} &= \left[ (6k+3)a + \frac{a}{2} + 3, (6k+4)a + \frac{a}{2} - 1 \right]_2, \\
A_{6k+4,k} &= [(6k+4)a, (6k+5)a]_2.
\end{aligned}$$

Therefore

$$\left[ (6k+3)a + \frac{a}{2} + 3, (6k+4)a + \frac{a}{2} \right] \cup \{(6k+4)a + \frac{a}{2} + 2\} \subseteq G$$

and consequently

$$\left[ (6k+4)a + \frac{a}{2} + 3, (6k+5)a + \frac{a}{2} \right] \cup \{(6k+5)a + \frac{a}{2} + 2\} \subseteq G.$$

Since

$$(6k+5)a + \frac{a}{2} + 1 = (6k+4)a - 1 + c,$$

we conclude that

$$\text{Ap}(G, a) \subseteq \left[ a, (6k+5)a + \frac{a}{2} + 2 \right].$$

Hence  $C \subseteq G$ .

Now let  $x \in G$ .

If  $x \geq (6k+4)a + \frac{a}{2} + 3$ , then  $x \in C$ .

Conversely we have that

$$x = (q + 3v + \varepsilon)a + \varepsilon \cdot \frac{a}{2} + 6v + 3\varepsilon + 2r$$

for some  $\{q, r, v\} \subseteq \mathbb{N}$ ,  $\varepsilon \in \{0, 1\}$  and  $0 \leq r \leq q$ .

If  $\varepsilon = 0$ , then  $x = ia + 2h$ , where

$$\begin{aligned} i &:= q + 3v, \\ h &:= 3v + r, \end{aligned}$$

with  $0 \leq h \leq 3v + q = i \leq 6k + 4$ . Hence  $x \in A_{i,k}$ .

If  $\varepsilon = 1$ , then  $x = ia + \frac{a}{2} + 3 + 2h$ , where

$$\begin{aligned} i &:= q + 3v + 1, \\ h &:= 3v + r, \end{aligned}$$

with  $0 \leq h \leq 3v + q = i - 1 \leq 6k + 3$ . Hence  $x \in B_{i,k}$ .

(3) All inequalities follow from the definition of the sets.

(4) First we notice that

$$\begin{aligned} |U_{i,k} \cap G| &= i + 1, \\ |V_{i,k} \cap G| &= i, \\ |(U_{i,k} \cup V_{i,k}) \cap G| &= 2i + 1, \end{aligned}$$

for any  $i \in [1, 3k + 1]$ .

Now we take a set of consecutive integers  $\{i_1, i_2, i_3\}$  such that  $1 \leq i_1 < i_2 < i_3 \leq 3k + 1$  and  $i_j \equiv j \pmod{3}$  for any  $j \in \{1, 2, 3\}$ . For any such an index  $j$  we have that

$$\begin{aligned} A_{i_j,k} &= U_{i_j,k} \cap G, \\ B_{i_j,k} &= V_{i_j,k} \cap G. \end{aligned}$$

If  $j = 1$ , then  $|(U_{i_1,k} \cup V_{i_1,k}) \cap G| \equiv 0 \pmod{3}$  and the sequence formed by the greatest two elements of  $A_{i_1,k}$  and the smallest element of  $B_{i_1,k}$  is a 3-permutation according to the definition of such sets. The remaining elements of  $\overline{g \cap A_{i_1,k}}$  and  $\overline{g \cap B_{i_1,k}}$  can be obtained through concatenations of 3-permutations. We notice that this argument applies in particular when  $i = 3k + 1$ .

We can also check that the sequence formed by the greatest two elements of  $B_{i_2,k}$  and the smallest element of  $A_{i_3,k}$  is a 3-permutation. The sequences of the remaining elements of  $g$  belonging to  $A_{i_2,k}$ ,  $B_{i_2,k}$ ,  $A_{i_3,k}$  and  $B_{i_3,k}$  are obtained (modulo 3) through concatenations of 3-permutations too.

Now we notice that

$$\begin{aligned} |U_{3k+2,k} \cap G| &= 3k + 3 \equiv 0 \pmod{3}, \\ |V_{3k+2,k} \cap G| &= \frac{a}{4} - 1 \equiv 1 \pmod{3}. \end{aligned}$$

As regards the indices  $i \in [3k + 3, 6k + 3]$ , we have that

$$\begin{aligned} |U_{i,k} \cap G| &= \left(\frac{a}{4} + 2\right) + \left(i - \frac{a}{4}\right) = 2 + i, \\ |V_{i,k} \cap G| &= \left(\frac{a}{4} - 1\right) + \left(i - \frac{a}{4} + 1\right) = i - 2. \end{aligned}$$

In particular we notice that

$$|(U_{i,k} \cup V_{i,k}) \cap G| \equiv 2i \pmod{3}$$

for any  $i \in [3k + 2, 6k + 3]$ .

We have that

$$\begin{aligned} U_{3k+2,k} \cap G &= \left[ ia, ia + \frac{a}{2} \right]_2, \\ V_{3k+2,k} \cap G &= \left[ ia + \frac{a}{2} + 3, (i+1)a - 1 \right]_2. \end{aligned}$$

The sequence formed by the elements of

$$((U_{3k+2,k} \cup V_{3k+2,k}) \cap G) \setminus \{(3k+3)a - 1\}$$

is given (modulo 3) by a concatenation of 3-permutations. Moreover the sequence formed by  $(3k+3)a - 1$  and the two smallest elements of  $U_{3k+3,k} \cap G$  is (modulo 3) a 3-permutation too, since it reads as follows (modulo 3):

$$((3k+3)a - 1, (3k+3)a, (3k+3)a + 1).$$

Now let  $\{i_2, i_0, i_1\}$  be a set of three consecutive integers such that  $3k+2 \leq i_2 < i_0 < i_1 \leq 6k+1$  and  $i_j \equiv j \pmod{3}$  for any  $j$ . Then

$$\sum_{j=0}^2 |((U_{i_j} \cap G) \cup (V_{i_j} \cap G))| \equiv 0 \pmod{3}.$$

Moreover the sequence of elements in

$$\cup_{j=0}^2 ((U_{i_j} \cap G) \cup (V_{i_j} \cap G))$$

is given (modulo 3) by concatenations of 3-permutations. We analyse in detail just the case  $j = 2$ . Since we have already considered the case  $i_j = 3k+2$ , we can suppose that  $i_j > 3k+2$ .

The elements of  $U_{i_2,k} \cap G$  are given by the following union of intervals:

$$\left[ i_2 a, (i_2 - 1)a + \frac{a}{2} + 2i_2 - 1 \right] \cup \left[ (i_2 - 1)a + \frac{a}{2} + 2i_2, i_2 a + \frac{a}{2} + 2 \right]_2.$$

We notice that the sequence (modulo 3) formed by the elements in

$$\left[ i_2 a, (i_2 - 1)a + \frac{a}{2} + 2i_2 - 1 \right]$$

is given by a concatenation of 3-permutations. Moreover

$$\left| \left[ (i_2 - 1)a + \frac{a}{2} + 2i_2, i_2 a + \frac{a}{2} + 2 \right]_2 \right| \equiv 1 \pmod{3}.$$

The sequence formed by the greatest element of  $U_{i_2,k} \cap G$  and the two smallest elements in  $V_{i_2,k} \cap G$  reads as follows (modulo 3):

$$\left( \overline{i_2 a + \frac{a}{2} + 2}, \overline{i_2 a + \frac{a}{2} + 3}, \overline{i_2 a + \frac{a}{2} + 4} \right) = (\bar{1}, \bar{2}, \bar{0}).$$

In a similar way we can check that the sequence (modulo 3) of elements in

$$(V_{i_2,k} \cap G) \setminus \{(i_2 + 1)a - 1\}$$

is given by a concatenation of 3-permutations.

Finally we notice that if

$$(i_2, i_0, i_1) = (6k+2, 6k+3, 6k+4),$$

then

$$|\cup_{j \in \{2,0\}} ((U_{i_j,k} \cup V_{i_j,k}) \cap G)| \equiv 1 \pmod{3}.$$

The sequence formed by the greatest element of  $V_{6k+3,k} \cap G$  and the smallest two elements of  $U_{6k+4,k}$  reads (modulo 3) as a 3-permutation. The sequence of the elements in  $G$  greater than  $(6k+4)a+1$  is given by the elements in

$$\left[ (6k+4)a+2, (6k+4)a+\frac{a}{2}-3 \right] \cup \left[ (6k+4)a+\frac{a}{2}-1, \infty \right[.$$

The elements belonging to this latter union of intervals are given (modulo 3) by concatenations of 3-permutations.  $\square$

**Lemma 4.16.** *Let  $k$  be a positive integer,  $a := 12k+8$ ,  $b := a + \frac{a}{2} + 1$  and  $c := a + \frac{a}{2} + 3$ .*

*Let  $S := \{a, b, c\}$  and*

$$H_{16,k} := (\cup_{i=0}^{3k+2} (A_{i,k} \cup B_{i,k})) \cup (\cup_{j=0}^{3k+1} (C_{j,k} \cup D_{j,k})) \cup E,$$

*where  $A_{0,k} := \{0\}$ ,  $B_{0,k} := \emptyset$ , and*

$$A_{i,k} := \left[ ia, ia + 6 \left\lfloor \frac{i}{3} \right\rfloor \right]_2,$$

$$B_{i,k} := A_{i-1,k} + \{b, c\},$$

*for any  $i \in [1, 3k+2]$ , while*

$$C_{j,k} := \left[ (3k+3+j)a, (3k+3+j)a + 6 \left\lfloor \frac{j+1}{3} \right\rfloor \right]$$

$$\cup \left[ (3k+3+j)a + 6 \left\lfloor \frac{j+1}{3} \right\rfloor + 2, (3k+3+j)a + \frac{a}{2} \right]_2,$$

$$D_{j,k} := \left[ (3k+3+j)a + \frac{a}{2} + 1, (3k+3+j)a + \frac{a}{2} + 3 + 6 \left\lfloor \frac{j}{3} \right\rfloor \right]$$

$$\cup \left[ (3k+3+j)a + \frac{a}{2} + 5 + 6 \left\lfloor \frac{j}{3} \right\rfloor, (3k+4+j)a - 1 \right]_2,$$

*for any  $j \in [0, 3k+1]$  and*

$$E := \left[ (6k+4)a + \frac{a}{2} + 1, \infty \right[.$$

*Then the following hold.*

(1)  $x \in G$  if and only if

$$x = (s+q)a + q \cdot \frac{a}{2} + q + 2r,$$

*where  $\{q, r, s\} \subseteq \mathbb{N}$  with  $0 \leq r \leq q$ .*

(2)  $S \subseteq H_{16,k} \subseteq G$ .

(3) *The following inequalities hold:*

$$\begin{aligned} A_{i,k} &< B_{i,k} && \text{for any } i \in [1, 3k+2], \\ B_{i,k} &< A_{i+1,k} && \text{for any } i \in [1, 3k+1], \\ B_{3k+2,k} &< C_{0,k}, \\ C_{j,k} &< D_{j,k} && \text{for any } j \in [0, 3k+1], \\ D_{j,k} &< C_{j+1,k} && \text{for any } j \in [0, 3k], \\ D_{3k+1,k} &< E. \end{aligned}$$

- (4)  $H_{16,k}$  is a numerical semigroup.  
 (5)  $H_{16,k}$  is a 3-permutation semigroup.

*Proof.* (1) We have that  $x \in G$  if and only if

$$x = sa + t \left( a + \frac{a}{2} + 1 \right) + u \left( a + \frac{a}{2} + 3 \right)$$

for some  $\{s, t, u\} \subseteq \mathbb{N}$ . According to Lemma 2.2 we can write

$$x = sa + q \left( a + \frac{a}{2} + 1 \right) + 2r = (s + q)a + q \cdot \frac{a}{2} + q + 2r$$

for some  $\{q, r\} \subseteq \mathbb{N}$  with  $0 \leq r \leq q$ .

- (2) First we notice that  $S \subseteq (A_{1,k} \cup B_{1,k})$ .

Let  $x \in A_{i,k}$  for some  $i \in [1, 3k + 2]$ . Then

$$x = ia + 2h$$

with  $0 \leq h \leq 3 \lfloor \frac{i}{3} \rfloor$ . We can write

$$2h = 2\tilde{h} + 2r$$

for some  $\{\tilde{h}, r\} \subseteq \mathbb{N}$  such that

$$0 \leq \tilde{h} \leq \left\lfloor \frac{i}{3} \right\rfloor \quad \text{and} \quad 0 \leq r \leq 2\tilde{h}.$$

If we set

$$s := i - 3\tilde{h},$$

$$q := 2\tilde{h},$$

then

$$x = (s + q)a + q \cdot \frac{a}{2} + q + 2r \in G.$$

If  $x \in B_{i,k}$  for some  $i \in [1, 3k + 2]$ , then  $x \in G$  by definition of the sets  $B_{i,k}$ .

Now we consider the sets  $C_{j,k}$ .

First we notice that

$$C_{0,k} \subseteq (A_{3k,k} + \{3a, 2b, 2c\}),$$

$$C_{1,k} = C_{0,k} + \{a\},$$

$$D_{0,k} \subseteq (B_{3k+2,k} + \{a\}) \cup (B_{3k+1,k} + \{c\}),$$

$$D_{1,k} = D_{0,k} + \{a\}.$$

Then we have that

$$C_{2,k} \subseteq (C_{1,k} + \{a\}) \cup (C_{0,k} + \{b, c\}) \cup (D_{0,k} + \{c\}),$$

$$D_{2,k} = D_{1,k} + \{a\}.$$

Finally, for any  $j \in [3, 3k + 1]$  the following hold.

- If  $j \equiv 0 \pmod{3}$ , then

$$C_{j,k} = C_{j-1,k} + \{a\},$$

$$D_{j,k} \subseteq D_{j-3,k} + \{3a, 2b, 2c\}.$$

- If  $j \equiv 1 \pmod{3}$ , then

$$C_{j,k} = C_{j-1,k} + \{a\},$$

$$D_{j,k} = D_{j-1,k} + \{a\}.$$



- If  $j \equiv 2 \pmod{3}$ , then

$$\begin{aligned} C_{j,k} &\subseteq C_{j-3,k} + \{3a, 2b, 2c\}, \\ D_{j,k} &= D_{j-1,k} + \{a\}. \end{aligned}$$

Now we notice that

$$D_{3k+1,k} = \left[ (6k+4)a + \frac{a}{2} + 1, (6k+5)a - 1 \right]$$

and

$$[(6k+5)a, (6k+6)a] \subseteq C_{3k-1,k} + \{3a, 2b, 2c\}.$$

Hence  $\text{Ap}(G, a) \subseteq [a, (6k+6)a]$  and  $E \subseteq G$ .

- (3) All inequalities follow immediately from the definition of the sets.
- (4) We notice that  $H_{16,k}$  is a co-finite subset of  $\mathbb{N}$ .

Now we take  $\{x, y\} \subseteq H_{16,k}$  and show that  $x + y \in H_{16,k}$  dealing with different cases.

- *Case 1:*  $x \in E$  or  $y \in E$ . Then  $x + y \in E$ .
- *Case 2:*  $\{x, y\} \subseteq [(3k+3)a, \infty[$ . Then  $x + y \in E$ .
- *Case 3:*  $x < (3k+3)a$  and  $y < (6k+4)a + \frac{a}{2} + 1$ .

We have that

$$x \in A_{i_1,k} \cup B_{i_1,k}$$

for some  $i_1 \in [0, 3k+2]$ . We analyse in detail just the case  $x \in A_{i_1,k}$ .

If  $x + y \in E$ , then we are done.

Now we suppose that  $x + y \notin E$  and consider some subcases.

- If  $y \in A_{i_2,k}$  for some  $i_2 \in [0, 3k+2]$ , then

$$x + y \in A_{i_1+i_2,k}$$

if  $i_1 + i_2 \leq 3k+2$ , else

$$x + y \in C_{i_1+i_2-(3k+3),k} \cup D_{i_1+i_2-(3k+3),k}.$$

- If  $y \in B_{i_2,k}$  for some  $i_2 \in [0, 3k+2]$ , then

$$x + y \in B_{i_1+i_2,k}$$

if  $i_1 + i_2 \leq 3k+2$ , else

$$x + y \in D_{i_1+i_2-(3k+3),k} \cup C_{i_1+i_2+1-(3k+3),k}.$$

- If  $y \in C_{j,k}$  for some  $j \in [0, 3k+1]$ , then

$$x + y \in C_{i+j,k} \cup D_{i+j,k}.$$

- If  $y \in D_{j,k}$  for some  $j \in [0, 3k+1]$ , then

$$x + y \in D_{i+j,k} \cup C_{i+j+1,k}.$$

- (5) For any  $i \in [1, 3k+2]$  we have that

$$|A_{i,k}| = 3 \left\lfloor \frac{i}{3} \right\rfloor + 1 \equiv 1 \pmod{3},$$

$$|B_{i,k}| = 3 \left\lfloor \frac{i-1}{3} \right\rfloor + 2 \equiv 2 \pmod{3}.$$

Therefore  $|A_{i,k} \cup B_{i,k}| \equiv 0 \pmod{3}$ .

The sequence formed by the greatest element of  $A_{i,k}$  and the two smallest elements of  $B_{i,k}$  reads as follows (modulo 3):

$$\left( \overline{ia + 6 \left\lfloor \frac{i}{3} \right\rfloor}, \overline{(i-1)a + b}, \overline{(i-1)a + b + 2} \right) = (\overline{2i}, \overline{2i+2}, \overline{2i+1}).$$

All the other elements of  $\overline{g \cap A_{i,k}}$  and  $\overline{g \cap B_{i,k}}$  can be obtained via concatenations of 3-permutations.

As regards the sets  $C_{j,k}$  and  $D_{j,k}$ , we have that

$$\begin{aligned} |C_{j,k}| &\equiv 0 \pmod{3}, \\ |D_{j,k}| &\equiv 0 \pmod{3}. \end{aligned}$$

Moreover, the elements of  $\overline{g \cap C_{j,k}}$  and  $\overline{g \cap D_{j,k}}$  are given by concatenations of 3-permutations.

Therefore  $H_{16,k}$  is a 3-permutation semigroup. □

## 5. OPEN QUESTIONS

**5.1. Towards the classification of 3-permutation semigroups.** In Section 4 we have constructed 16 families of 3-permutation semigroups.

We notice that if  $G$  is a 3-permutation semigroup and  $g_1 < g_2 < g_3$  are the three smallest positive elements of  $G$ , then

$$\begin{aligned} g_2 &\leq 2g_1, \\ g_3 &\leq 3g_1. \end{aligned}$$

We have written a GAP function `persgp(k,m,n)`, which returns the  $k$  generators  $g_1 < \dots < g_k$  of a  $k$ -permutation semigroup such that  $m \leq g_1 < \dots < g_k \leq n$ .

In accordance with the remark above, we can find all 3-permutation semigroups of a fixed multiplicity  $M > 0$  running the function `persgp(k,m,n)` with parameters  $k = 3$ ,  $m \leq M$  and  $n \geq 3M$ .

For example, we can get the list of all 3-permutation semigroups having multiplicity not greater than 11 running `persgp(3,1,33)`. The generating set of any such semigroup is one of the sets in Table 1.

In Table 2 we list all generating sets of 3-permutation semigroups having multiplicity between 12 and 35. For any such a semigroup we also specify the family it belongs to. Such a list has been obtained running `persgp(3,12,105)`.

Table 1
$S$
$\{1, 2, 3\}$
$\{2, 3, 4\}$
$\{3, 4, 5\}$
$\{4, 5, 6\}$
$\{5, 6, 7\}$
$\{5, 7, 9\}$
$\{6, 7, 11\}$
$\{7, 8, 12\}$
$\{7, 9, 11\}$
$\{7, 11, 12\}$
$\{8, 9, 10\}$
$\{8, 10, 15\}$
$\{8, 13, 15\}$
$\{9, 10, 17\}$
$\{9, 11, 16\}$
$\{10, 11, 15\}$
$\{10, 17, 18\}$
$\{11, 12, 13\}$
$\{11, 13, 21\}$
$\{11, 15, 16\}$
$\{11, 16, 18\}$
$\{11, 19, 21\}$

Table 2		
$S$	$S$	Family
$\{12, 13, 23\}$	$\{24, 25, 47\}$	$H_{1,k}$
$\{13, 14, 21\}$	$\{25, 26, 39\}$	$H_{2,k}$
$\{13, 15, 17\}$	$\{25, 27, 29\}$	$H_{4,k}$
$\{13, 20, 21\}$	$\{25, 38, 39\}$	$H_{3,k}$
$\{13, 20, 24\}$	$\{25, 44, 48\}$	$H_{5,k}$
$\{13, 23, 24\}$	$\{25, 47, 48\}$	$H_{6,k}$
$\{14, 15, 16\}$	$\{26, 27, 28\}$	$H_{7,k}$
$\{14, 16, 21\}$	$\{26, 28, 39\}$	$H_{8,k}$
$\{14, 25, 27\}$	$\{26, 49, 51\}$	$H_{9,k}$
$\{15, 16, 29\}$	$\{27, 28, 53\}$	$H_{1,k}$
$\{15, 17, 28\}$	$\{27, 29, 52\}$	$H_{10,k}$
$\{16, 17, 24\}$	$\{28, 29, 42\}$	$H_{11,k}$
$\{16, 21, 23\}$	$\{28, 39, 41\}$	$H_{12,k}$
$\{16, 29, 30\}$	$\{28, 53, 54\}$	$H_{6,k}$
$\{17, 18, 19\}$	$\{29, 30, 31\}$	$H_{7,k}$
$\{17, 24, 25\}$	$\{29, 42, 43\}$	$H_{13,k}$
$\{17, 28, 30\}$	$\{29, 52, 54\}$	$H_{14,k}$
$\{17, 31, 33\}$	$\{29, 55, 57\}$	$H_{9,k}$
$\{18, 19, 35\}$	$\{30, 31, 59\}$	$H_{1,k}$
$\{19, 20, 30\}$	$\{31, 32, 48\}$	$H_{2,k}$
$\{19, 21, 23\}$	$\{31, 33, 35\}$	$H_{4,k}$
$\{19, 29, 30\}$	$\{31, 47, 48\}$	$H_{3,k}$
$\{19, 32, 36\}$	$\{31, 56, 60\}$	$H_{5,k}$
$\{19, 35, 36\}$	$\{31, 59, 60\}$	$H_{6,k}$
$\{20, 21, 22\}$	$\{32, 33, 34\}$	$H_{7,k}$
$\{20, 22, 33\}$	$\{32, 34, 51\}$	$H_{15,k}$
$\{20, 31, 33\}$	$\{32, 49, 51\}$	$H_{16,k}$
$\{20, 37, 39\}$	$\{32, 61, 63\}$	$H_{9,k}$
$\{21, 22, 41\}$	$\{33, 34, 65\}$	$H_{1,k}$
$\{21, 23, 40\}$	$\{33, 35, 64\}$	$H_{10,k}$
$\{22, 23, 33\}$	$\{34, 35, 51\}$	$H_{11,k}$
$\{22, 41, 42\}$	$\{34, 65, 66\}$	$H_{6,k}$
$\{23, 24, 25\}$	$\{35, 36, 37\}$	$H_{7,k}$
$\{23, 33, 34\}$	$\{35, 51, 52\}$	$H_{13,k}$
$\{23, 40, 42\}$	$\{35, 64, 66\}$	$H_{14,k}$
$\{23, 43, 45\}$	$\{35, 67, 69\}$	$H_{9,k}$

The data above and other experimental data for 3-numerical semigroups seem to support the following conjecture.

**Conjecture 5.1.** *Any 3-numerical semigroup having multiplicity at least 12 belongs to one of the family  $H_{1,k} - H_{16,k}$  of Section 4.*

**5.2. On  $n$ -permutation semigroups.** In Lemma 5.2 we construct a family of  $n$ -permutation semigroups for any  $n \geq 3$ . We notice in passing that the 3-permutation

semigroups of the family  $H_{6,k}$  belong to this more general family of  $n$ -permutation semigroups.

The experimental evidence suggests that many more  $n$ -numerical semigroups exist, but we think that a classification of all of them is (at the moment) out of reach.

**Lemma 5.2.** *Let  $n \geq 3$  be an integer,  $k$  a positive integer and  $a := nk + 1$ . Let  $S := \{a\} \cup [2a - n, 2a - 2]$  and  $G := \langle S \rangle$ .*

*Let  $H := \{0\} \cup (\cup_{i=0}^{k-1} (A_{i,k} \cup B_{i,k})) \cup [2ka, \infty[$ , where*

$$A_{i,k} := \{(2i+1)a\} \cup [(2i+2)a - (i+1)n, (2i+2)a - 2],$$

$$B_{i,k} := \{(2i+2)a\} \cup [(2i+3)a - (i+1)n, (2i+3)a - 2],$$

*for any  $i \in [0, k-1]$ .*

*Then the following hold.*

- (1)  *$H$  is a numerical semigroup.*
- (2)  *$G = H$ .*
- (3)  *$H$  is a  $n$ -permutation semigroup.*

*Proof.* (1) Let  $\{i_1, i_2\} \subseteq \mathbb{N}$  and  $i_3 := i_1 + i_2 + 1$ . If

$$x \in A_{i_1,k} \cup B_{i_2,k},$$

$$y \in A_{i_2,k} \cup B_{i_2,k},$$

then  $x+y$  belongs to one of the sets in rows 2–3, columns 2–3 of the following table.

	$A_{i_2,k}$	$B_{i_2,k}$
$A_{i_1,k}$	$A_{i_3,k} \cup B_{i_3-1,k}$	$A_{i_3,k} \cup B_{i_3,k}$
$B_{i_1,k}$		$A_{i_3+1,k} \cup B_{i_3,k}$

Since  $H$  is co-finite, we conclude that  $H$  is a numerical semigroup.

(2) First we notice that  $S \subseteq H$ .

Now we prove that  $H \subseteq G$ , discussing separately some cases.

- *Case 1:  $A_{i,k} \subseteq G$  for any  $i \in [0, k-1]$ .*

For any  $j \in \mathbb{N}$  we have that  $ja \in G$ .

Moreover, for any  $i \in [0, k-1]$  we have that

$$[(2i+2)a - (i+1)n, (2i+2)a - (2i+2)] \subseteq \langle [2a - n, 2a - 2] \rangle$$

according to Lemma 2.2.

We prove by induction on  $i \in [0, k-1]$  that

$$(2i+2)a - j \in G$$

for any  $j \in [2, 2i+1]$ .

If  $i = 0$ , then there is nothing to prove.

If  $i > 0$ , then

$$(2i+2)a - j = (2(i-1)+2)a - j + 2a.$$

Since  $j \leq 2i+1 \leq 3i \leq ni$ , we have that

$$(2(i-1)+2)a - j \geq (2(i-1)+2)a - in.$$

Therefore  $(2(i-1)+2)a - j \in A_{i-1,k}$ . Since  $A_{i-1,k} \subseteq G$  by inductive hypothesis, we conclude that  $(2i+2)a - j \in G$ .

- *Case 2:*  $B_{i,k} \subseteq G$  for any  $i \in [0, k-1]$ .  
We notice that  $B_{i,k} = A_{i,k} + \{a\}$  for any  $i$ . Hence  $B_{i,k} \subseteq G$  for any  $i$ .
- *Case 3:*  $[2ka, \infty[ \subseteq G$ .

First we notice that

$$\begin{aligned} A_{k-1,k} &= [(2k-1)a, 2ka-2], \\ B_{k-1,k} &= [2ka, (2k+1)a-2]. \end{aligned}$$

Then we observe that

$$(2k+1)a-1 = (2k-1)a+1 + (2a-2) \subseteq A_{k-1,k} + A_{0,k}.$$

Therefore  $[2ka, (2k+1)a-1] \subseteq G$ , namely  $\text{Ap}(G, a) \subseteq [a, (2k+1)a-1]$ .

Hence  $[2ka, \infty[ \subseteq G$ .

(3) We notice that

$$|A_{i,k}| = |B_{i,k}| = (i+1)n$$

for any  $i \in [0, k-1]$ , namely  $n$  divides the cardinality of any set  $A_{i,k}$  and  $B_{i,k}$ .

Since the elements of the sets  $A_{i,k}$  and  $B_{i,k}$  are obtained through concatenations of  $n$ -permutations, we conclude that  $H$  is a  $n$ -permutation semigroup.  $\square$

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